Optimal Step-stress Tests for Exponential Failure Data under Progressive Type-I Censoring

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Abstract
A k-step-stress accelerated life-testing is considered with an equal step duration τ. For small to moderate sample sizes, a practical modification is suggested in order to guarantee a feasible k-step-stress test under progressive Type-I censoring, and the optimal τ is determined under this model. Next, we discuss the determination of optimal τ under the condition that the step-stress test proceeds to the k-th stress level, and the efficiency of this conditional inference is compared to that of the previous case. In all cases considered, censoring is allowed at each point of stress change. The determination of optimal τ is discussed under C-optimality, D-optimality, and A-optimality criteria. We investigate the case of progressively Type-I right censored data from an exponential distribution with a single stress variable.

Key Words: Accelerated life-testing, A-optimality, Cumulative exposure model, C-optimality, D-optimality, Fisher information, Maximum likelihood estimation, Progressive Type-I censoring

1. Introduction

As the products become highly reliable with substantially long life-spans, time-consuming and expensive tests are frequently required to collect enough failure data, which are necessary to draw inference about the relationship of lifetime with external stress variables ($i.e.$, covariates). In such situations, standard life-testing methods are not suitable. This difficulty is overcome by accelerated life-test (ALT) wherein the units are subjected to higher stress levels in order to cause rapid failures. ALT allows the experimenter to apply more severe stresses to obtain information on the parameters of lifetime distributions more quickly than would be possible under normal operating conditions. Some key references in the area of accelerated testing include Nelson (1990), Meeker and Escobar (1998), and Bagdonavicius and Nikulin (2002). A special class of ALT is the step-stress testing which allows the experimenter to gradually increase the stress levels at some fixed time points during the experiment for maximal flexibility and adjustability.

In addition, due to time constraints and also to reduce the cost of experimentation, censoring is also unavoidable in practice in such a reliability test. Censored data arise when experiments involving lifetimes of machines or products have to be terminated before complete sampling, and progressive Type I right-censored samples are observed when a pre-specified number or proportion of live test units are progressively removed from the experiment at the end of each testing stage. The importance of progressive censoring (PC) lies in its efficient exploitation of the available resources compared to the traditional sampling. Withdrawn unfailed testing units can be used in other experiments in the same or at a different facility; see, for example, Cohen (1991), Balakrishnan and Aggarwala (2000), and Viveros and Balakrishnan (1994).

The main focus of this article is to build a feasible ALT model combined with PC for a small to moderate sample size, and then to investigate the choice of optimal change points of the stress levels with or without the condition that the life-test proceeds to the last stage of stress. A practical modification is suggested to the asymptotic model discussed by

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Gouno, Sen and Balakrishnan (2004) for a feasible step-stress analysis under a PC scheme with an arbitrary number of stress levels; see also Han *et al.* (2006) for some related comments. Here, we consider the equi-spaced step with τ denoting the duration of each testing stage. Since we must decide upon the length of an inspection interval, this setup for a k-step-stress test seems reasonable and pragmatic. Using three different optimality criteria ($viz.$, variance, determinant and trace), the efficiency of the conditional approach to the optimality problem is also discussed, and a comparison of the numerical results from the asymptotic and the modified models is presented as well.

2. Model description and MLEs

PC is a generalized form of censoring which includes the conventional right censoring as a special case. To describe the step-stress testing procedure implemented with a popular form of PC, called progressive Type-I censoring, let us first define $x_1 < x_2 < \ldots < x_k$ to be the ordered stress levels to be used in the test. Then, for $i = 1, 2, \ldots, k$, let n_i denote the number of units failed at stress level x_i ($i.e.$, in time interval $[(i-1)\tau, i\tau)$) and $y_{i,l}$ denote the l-th ordered failure time of n_i units at x_i, $l = 1, 2, \ldots, n_i$ while c_i denotes the number of units censored at time $i\tau$. Furthermore, let N_i denote the number of units operating and remaining on test at the start of stress level x_i $\left(viz., N_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} c_j\right)$.

Under this setup, a step-stress test with an equal step duration τ proceeds as follows. A total of $N_1 \equiv n$ test units is initially placed at stress level x_1 and tested until time τ at which point the stress is changed to x_2 and c_1 live items are arbitrarily withdrawn from the test. The test is continued on $N_2 = n - n_1 - c_1$ units until time 2τ, when the stress is changed to x_3 and c_2 items are withdrawn from the test, and so on. Finally, at time $k\tau$, all the surviving items are withdrawn, thereby terminating the life-test. Note that since $n \equiv \sum_{i=1}^{k}(n_i + c_i)$, the number of surviving items at time $k\tau$ is $c_k = n - \sum_{i=1}^{k} n_i - \sum_{i=1}^{k-1} c_i = N_k - n_k$. Obviously, when there is no intermediate censoring ($viz.$, $c_1 = c_2 = \cdots = c_{k-1} = 0$), this situation corresponds to the k-level step-stress testing under Type-I right censoring as a special case. Now, the following assumptions are crucial for constructing subsequent step-stress models.

Assumptions **:**

(i) A cumulative exposure model holds;

(ii) For any stress level, the lifetime of a test unit follows an exponential distribution;

(iii) At stress level x_i, the mean time to failure (MTTF) of a test unit, θ_i, is a log-linear function of stress given by

$$\log\theta_i = \alpha + \beta x_i, \tag{2.1}$$

where the regression parameters α and β are unknown and need to be estimated.

Under the assumptions (i) and (ii), the probability density function (PDF) of a test unit is

$$f(t) = f_i(t-(i-1)\tau)\prod_{j=1}^{i-1} S_j(\tau) \qquad \text{if } \begin{cases} (i-1)\tau \le t \le i\tau & \text{for } i = 1, 2, \ldots, k-1 \\ (k-1)\tau \le t < \infty & \text{for } i = k \end{cases} \tag{2.2}$$

where $f_i(t) = \dfrac{1}{\theta_i}\exp\left(-\dfrac{t}{\theta_i}\right)$. The corresponding cumulative distribution function (CDF) is then given by

$$\begin{aligned} F(t) &= 1 - \left[\prod_{j=1}^{i-1} S_j(\tau)\right] S_i(t-(i-1)\tau) \\ &\qquad \text{if} \begin{cases} (i-1)\tau \le t \le i\tau & \text{for } i = 1,2,\ldots,k-1 \\ (k-1)\tau \le t < \infty & \text{for } i = k \end{cases}, \end{aligned} \tag{2.3}$$

where

$$F_i(t) = 1 - S_i(t) = 1 - \exp\left(-\frac{t}{\theta_i}\right).$$

No notational distinction will be made in this article between the random variables and their corresponding realizations. Also, we adopt the usual conventions that $\sum_{j=m}^{m-1} a_j \equiv 0$ and $\prod_{j=m}^{m-1} a_j \equiv 1$. Then, using (2.2) and (2.3), the joint probability density function (JPDF) of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k})$ with $\mathbf{y_i} = (y_{i,1}, y_{i,2}, \ldots, y_{i,n_i})$ is obtained as

$$f_J(\mathbf{y}, \mathbf{n}) = \left[\prod_{i=1}^{k} \frac{N_i!}{(N_i - n_i)!}\right]\left[\prod_{i=1}^{k} \theta_i^{-n_i}\right] \exp\left(-\sum_{i=1}^{k} \frac{U_i}{\theta_i}\right), \tag{2.4}$$

where

$$U_i = \sum_{j=1}^{n_i} (y_{i,j} - (i-1)\tau) + (N_i - n_i)\tau, \qquad i = 1,2,\ldots,k. \tag{2.5}$$

Note that U_i in (2.5) is precisely the *Total Time on Test* statistic at stress level x_i. Now, using (2.4) and assumption (iii), the log-likelihood function of (α, β) can be written as

$$l(\alpha, \beta) = -\alpha \sum_{i=1}^{k} n_i - \beta \sum_{i=1}^{k} n_i x_i - \sum_{i=1}^{k} U_i \exp[-(\alpha + \beta x_i)]. \tag{2.6}$$

After differentiating $l(\alpha, \beta)$ with respect to α and β, we obtain the likelihood equations as

$$0 = \frac{\partial}{\partial \alpha} l(\alpha, \beta) = -\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} U_i \exp[-(\alpha + \beta x_i)], \tag{2.7}$$

$$0 = \frac{\partial}{\partial \beta} l(\alpha, \beta) = -\sum_{i=1}^{k} n_i x_i + \sum_{i=1}^{k} U_i x_i \exp[-(\alpha + \beta x_i)]. \tag{2.8}$$

The MLEs $\hat{\alpha}$ and $\hat{\beta}$ are then obtained as simultaneous solutions to the following two equations:

$$\hat{\alpha} = \log\left(\frac{\sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i)}{\sum_{i=1}^{k} n_i}\right), \tag{2.9}$$

$$\left[\sum_{i=1}^{k} n_i\right]\left[\sum_{i=1}^{k} U_i x_i \exp(-\hat{\beta} x_i)\right] - \sum_{i=1}^{k} n_i x_i \sum_{i=1}^{k} U_i \exp(-\hat{\beta} x_i) = 0. \tag{2.10}$$

As shown above, $\hat{\alpha}$ and $\hat{\beta}$ are non-linear functions of random quantities, which make it virtually impossible to find their exact marginal/joint distributions for exact inference. Thus, statistical inference with these MLEs are based on the asymptotic distributional result that the vector $(\hat{\alpha}, \hat{\beta})$ is approximately distributed as a bivariate normal with mean vector (α, β) and variance-covariance matrix $[\mathbf{I}_n(\alpha, \beta)]^{-1}$, where $\mathbf{I}_n(\alpha, \beta)$ is the Fisher information.

3. k-level step-stress test under progressive censoring in small samples

In a reliability experiment, the sample size is usually small and there might be severe censoring due to various reasons including the budgetary constraints and facility requirements. Under such circumstances, the assumptions made by Gouno, Sen and Balakrishnan (2004) are violated and therefore, a modification is required to their proposed model so that a feasible PC scheme can be guaranteed. One such modification which can be entertained in practice is to decide on a fixed proportion of unfailed items to be removed at the end of each stage, rather than to decide on a global proportion over the initial sample size. Since the number of live units at the end of each stage before censoring takes place is random, the proposed change essentially makes the number of progressively censored units also random.

In order to revise the model according to the proposed modification, we first define a vector of proportions

$$\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*, \ldots, \pi_{k-1}^*),$$

where $0 \leq \pi_i^* < 1$ for $i = 1, 2, \ldots, k-1$. Note that $\boldsymbol{\pi}^*$ is composed of fixed constants defining the proportion of surviving items to be censored at each stress transition. Thus, $\pi_i = c_i/n$, the overall censoring proportion at the i-th stage defined over the total number of testing units is distinguished from π_i^*. Since all the remaining items are withdrawn from the test at the end of stress level x_k, one could also state $\pi_k^* = 1$. In this setting, the number of censored items at the end of stress level x_i is

$$c_i = \Upsilon((N_i - n_i)\pi_i^*) \qquad \text{for } i = 1, 2, \ldots, k-1, \tag{3.1}$$

where $\Upsilon(\cdot)$ is a discretizing function of one's choice, mapping its argument to a non-negative integer. $\Upsilon(\cdot)$ could be one of $round(\cdot)$, $floor(\cdot)$, $ceiling(\cdot)$ and $trunc(\cdot)$, for example. Since $0 \leq \pi_i^* < 1$, $0 \leq c_i \leq N_i - n_i$ for $i = 1, 2, \ldots, k-1$. When $c_i = N_i - n_i \geq 0$, the life-test terminates at the end of the i^*-th stage where i^* is the minimum of such i's satisfying $c_i = N_i - n_i$. Consequently, this results in $N_{i^*+1} = N_{i^*+2} = \cdots = N_k = 0$, $n_{i^*+1} = n_{i^*+2} = \cdots = n_k = 0$, and $c_{i^*+1} = c_{i^*+2} = \cdots = c_k = 0$ since $N_{i+1} = N_i - n_i - c_i$. Hence, under the proposed modification, we allow the life-test to terminate before reaching the last stress level x_k. We should also point out that $\mathbf{c} = (c_1, c_2, \ldots, c_{k-1})$ is random as well as $\boldsymbol{\pi} = \mathbf{c}/n = (\pi_1, \pi_2, \ldots, \pi_{k-1})$ under this setup. When $\boldsymbol{\pi}^* = (0, 0, \ldots, 0) = \mathbf{0_{k-1}}$, we have $\mathbf{c} = \mathbf{0_{k-1}}$ and $\boldsymbol{\pi} = \mathbf{0_{k-1}}$, and it is clear that this case corresponds to the case of a k-level step-stress testing under Type-I right censoring. In addition, if $c_k > 0$ or $n_k > 0$ (equivalently, $N_k = n_k + c_k > 0$), it implies that the life-test has proceeded onto the last stress level x_k.

The definition of c_i in (3.1) nevertheless complicates the derivation of distributions of associated random quantities. For simplicity, we shall assume in all subsequent derivations that

$$c_i = (N_i - n_i)\pi_i^* \qquad \text{for } i = 1, 2, \ldots, k-1, \tag{3.2}$$

as $\Upsilon((N_i - n_i)\pi_i^*) \approx (N_i - n_i)\pi_i^*$. Then, by using the following properties of the counts and order statistics, we can derive the expectation of N_i and also obtain the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$.

Properties **:**

(1) The random variable n_1 has a binomial distribution with parameters $(n, F_1(\tau))$. For $i = 2, 3, \ldots, k$, given $n_1, n_2, \ldots, n_{i-1}$, the random variable n_i has a binomial distribution with parameters $(N_i, F_i(\tau))$.

(2) Given $n_1, n_2, \ldots, n_i$, the random variables $(y_{i,j} - (i-1)\tau)$, $j = 1, 2, \ldots, n_i$, are distributed jointly as order statistics from a random sample of size n_i from a right-truncated exponential distribution with PDF $f_{i,\tau}(z) = \dfrac{f_i(z)}{F_i(\tau)}$ for $0 \leq z \leq \tau$ and $i = 1, 2, \ldots, k$.

Lemma 3.1. *For* $i = 1, 2, \ldots, k$*,*

$$E[N_i] = n \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*). \tag{3.3}$$

Theorem 3.1. *Under the proposed modification, the Fisher information matrix is*

$$\mathbf{I}_n(\alpha, \beta) = n \begin{pmatrix} \sum\limits_{i=1}^{k} A_i(\tau) & \sum\limits_{i=1}^{k} A_i(\tau) x_i \\ \sum\limits_{i=1}^{k} A_i(\tau) x_i & \sum\limits_{i=1}^{k} A_i(\tau) x_i^2 \end{pmatrix}, \tag{3.4}$$

where

$$A_i(\tau) = F_i(\tau) \prod_{j=1}^{i-1} S_j(\tau)(1 - \pi_j^*). \tag{3.5}$$

4. Optimality criteria and existence of optimal stress change point

We define different optimality criteria for determining an optimal stress duration τ. These objective functions are based on the Fisher information matrix $\mathbf{I}_n(\alpha, \beta)$ derived in the preceding section. Unlike $A_i(\tau)$ in Gouno, Sen and Balakrishnan (2004), $A_i(\tau)$ in (3.5) is positive for all $\tau > 0$. This, in turn, eliminates any disconcerting anomalies and ensures a positive determinant of $\mathbf{I}_n(\alpha, \beta)$ as well as a positive variance function. Since the censoring is performed based on the number of surviving units at the end of each stage, the case of censoring beyond what is available on test is completely avoided. Therefore, there is no restriction on the search region for the optimal τ in this case ($i.e.$, $\mathcal{C}_\tau = \{\tau : \tau > 0\}$).

4.1 *C-optimality*

In an ALT experiment, researchers often wish to estimate the parameters of interest with maximum precision and minimum variability possible. In the step-stress setting under consideration here, such a parameter of interest is the mean lifetime of a unit at the use-condition ($viz.$, θ_0). For this purpose, we consider an objective function from (3.4) as

$$\begin{aligned} \phi(\tau) &= n \text{ AVar}(\log \hat{\theta}_0) = n \text{ AVar}(\hat{\alpha} + \hat{\beta} x_0) \\ &= n\ (1, x_0) \mathbf{I}_n^{-1}(\alpha, \beta) \begin{pmatrix} 1 \\ x_0 \end{pmatrix} \\ &= \frac{2 \sum\limits_{i=1}^{k} A_i(\tau)(x_i - x_0)^2}{\sum\limits_{i=1}^{k} \sum\limits_{j=1}^{k} A_i(\tau) A_j(\tau)(x_i - x_j)^2}, \end{aligned} \tag{4.1}$$

where AVar stands for asymptotic variance and x_0 is the normal use-stress. The C-optimal τ ($viz.$, τ_C^*) is the one that minimizes $\phi(\tau)$ in (4.1). In the case of $k = 2$ ($i.e.$, the case of a simple step-stress test), the objective function in (4.1) under the C-optimality can be shown to reduce to

$$\begin{aligned} \phi(\tau) &= \frac{A_1(\tau)(x_1 - x_0)^2 + A_2(\tau)(x_2 - x_0)^2}{A_1(\tau)A_2(\tau)(x_2 - x_1)^2} \\ &= \frac{(1+\xi)^2}{A_1(\tau)} + \frac{\xi^2}{A_2(\tau)}, \end{aligned} \tag{4.2}$$

where $\xi = \dfrac{x_1 - x_0}{x_2 - x_1}$.

Theorem 4.1. *In the case of a simple step-stress test under progressive Type-I censoring, there exists an optimal step duration τ_C^* which is the unique solution of the equation $\phi'(\tau) = 0$.*

4.2 *D-optimality*

Another optimality criterion often used in planning ALT is based on the determinant of the Fisher information matrix, which is the same as the reciprocal of the determinant of the asymptotic variance-covariance matrix. Note that the overall volume of the asymptotic joint confidence region of (α, β) is proportional to $|\mathbf{I}_n^{-1}(\alpha, \beta)|^{1/2}$ at a fixed confidence level. In other words, it is inversely proportional to $|\mathbf{I}_n(\alpha, \beta)|^{1/2}$, the square root of the determinant of $\mathbf{I}_n(\alpha, \beta)$. Consequently, a larger value of $|\mathbf{I}_n(\alpha, \beta)|$ would correspond to a smaller asymptotic joint confidence ellipsoid of (α, β) and thus a higher joint precision of the estimators of α and β. Motivated by this, our second objective function is simply given by

$$\begin{aligned} \delta(\tau) &= n^{-2}|\mathbf{I}_n(\alpha, \beta)| \\ &= \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} A_i(\tau)A_j(\tau)(x_i - x_j)^2. \end{aligned} \tag{4.3}$$

The D-optimal τ ($viz.$, τ_D^*) is obtained by maximizing (4.3) for the maximal joint precision of $(\hat{\alpha}, \hat{\beta})$. For $k = 2$, the objective function (4.3) under the D-optimality reduces to

$$\delta(\tau) = A_1(\tau)A_2(\tau)(x_2 - x_1)^2. \tag{4.4}$$

Theorem 4.2. *In the case of a simple step-stress test under progressive Type-I censoring, the D-optimal stress change point τ_D^* is the solution of $A_1'(\tau)A_2(\tau) + A_1(\tau)A_2'(\tau) = 0$.*

4.3 *A-optimality*

Another optimality criterion considered in our study is based on the sum of marginal Fisher information terms of the parameters of the model. It is identical to the sum of the diagonal elements or trace of $\mathbf{I}_n(\alpha, \beta)$. Like the D-optimality, the A-optimality criterion is a general measure of the size of the Fisher information $\mathbf{I}_n(\alpha, \beta)$. The A-optimal τ ($viz.$, τ_A^*) maximizes the objective function defined by

$$\begin{aligned} a(\tau) &= \frac{1}{n} tr(\mathbf{I}_n(\alpha, \beta)) \\ &= \sum_{i=1}^{k} A_i(\tau) + \sum_{i=1}^{k} A_i(\tau)x_i^2 = \sum_{i=1}^{k} A_i(\tau)(1 + x_i^2). \end{aligned} \tag{4.5}$$

In the case of the simple step-stress test ($k = 2$), the objective function in (4.5) under the A-optimality simply becomes

$$a(\tau) = A_1(\tau)(1 + x_1^2) + A_2(\tau)(1 + x_2^2). \tag{4.6}$$

Theorem 4.3. *For the simple step-stress test under progressive Type-I censoring, the A-optimal stress change point is*

$$\tau_A^* = \theta_2 \log\left[\left(1 + \frac{\theta_1}{\theta_2}\right)(1 - Q_1^A)^{-1}\right], \qquad \textit{where} \qquad Q_1^A = \frac{1 + x_1^2}{(1 - \pi_1^*)(1 + x_2^2)},$$

and it exists when $\dfrac{x_2^2 - x_1^2}{1 + x_2^2} > \pi_1^*$. *Otherwise,* τ_A^* *does not exist.*

5. Conditional analysis of k-step-stress test under progressive censoring

Conditional analysis is particularly useful as we deal with a finite sample size. In this section, we adopt the notation and intermediate results from Sections 2 and 3, and formulate the distributional results required to tackle the problem of selecting an optimal stress duration using the conditional approach. Since the probability of premature termination of a life-test with a small sample size is much greater than the one with a large sample size, the derivation of the distributional results for a finite sample case is based on the condition that the planned censoring scheme is fully applied to the test. That is, there are enough testing units for censoring at each stress change. This condition is translated into the set $\{\mathbf{n} : N_2 > 0, N_3 > 0, \ldots, N_k > 0\}$ where $\{\mathbf{n} : N_i > 0\}$ defines a set of all the possible values $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ can take on satisfying the condition $N_i > 0$ ($i.e.$, successful censoring at time $(i-1)\tau$ for $i = 2, 3, \ldots, k$). As we find that

$$\{\mathbf{n} : N_k > 0\} \subset \{\mathbf{n} : N_{k-1} > 0\} \subset \cdots \subset \{\mathbf{n} : N_1 \equiv n > 0\} = \{\mathbf{n}\},$$

the condition simply yields $\{\mathbf{n} : N_2 > 0, N_3 > 0, \ldots, N_k > 0\} = \{\mathbf{n} : N_k > 0\}$. This proves that the condition of successful censoring at every stress level is equivalent to the condition of the test proceeding to the last stress level x_k. The probability of $N_k > 0$ is then easily obtained from the following lemma.

Lemma 5.1. *For* $i = 1, 2, \ldots, k-1$,

$$Pr(N_k = 0 | n_1, n_2, \ldots, n_{i-1}) = [H_i(\tau)]^{N_i}, \tag{5.1}$$

where

$$H_i(\tau) = \begin{cases} F_i(\tau) + S_i(\tau)[H_{i+1}(\tau)]^{1-\pi_i^*}, & \textit{for } i = 1, 2, \ldots, k-1 \\ 0, & \textit{for } i = k \end{cases}. \tag{5.2}$$

Corollary 5.1. *For k stress levels, the probability of a life-test proceeding to stress level x_k is*

$$Pr(N_k > 0) = 1 - [H_1(\tau)]^n. \tag{5.3}$$

Proof. Since $N_k \geq 0$, we obtain from Lemma 5.1,

$$Pr(N_k > 0) = 1 - Pr(N_k = 0) = 1 - [H_1(\tau)]^{N_1} = 1 - [H_1(\tau)]^n.$$

□

With the results above, the following lemma gives an expression for the expected number of failures observed at each stress level conditioned on $N_k > 0$. For this purpose, we denote $E_c[\,\cdot\,] = E[\,\cdot\,|N_k > 0]$ for the conditional expectation given $N_k > 0$.

Lemma 5.2. *For $i = 1, 2, \ldots, k$,*

$$E\Big[N_i[H_i(\tau)]^{N_i}\Big] = n[H_1(\tau)]^n \prod_{j=1}^{i-1}(1-\pi_j^*)\left(1-\frac{F_j(\tau)}{H_j(\tau)}\right). \tag{5.4}$$

Theorem 5.1. *For $i = 1, 2, \ldots, k$,*

$$E_c[n_i] = E[n_i|N_k > 0] = E[n_i]\frac{1-V_i(\tau)}{1-[H_1(\tau)]^n}, \tag{5.5}$$

where

$$V_i(\tau) = \begin{cases} \dfrac{[H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1}[H_{j+1}(\tau)]^{\pi_j^*}}, & \text{for } i = 1, 2, \ldots, k-1 \\ 0, & \text{for } i = k \end{cases} \tag{5.6}$$

and

$$E[n_i] = n\left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right]F_i(\tau). \tag{5.7}$$

We are now set to derive the Fisher information matrix $\mathbf{I}_n(\alpha,\beta)$ conditioned on $N_k > 0$ using the results presented above along with the following lemma.

Lemma 5.3. *For $i = 1, 2, \ldots, k$,*

$$E_c[N_i] = E[N_i|N_k > 0] = E[N_i]\left(\frac{1-H_i(\tau)V_i(\tau)}{1-[H_1(\tau)]^n}\right), \tag{5.8}$$

where $E[N_i]$ is as given in (3.3).

Theorem 5.2. *The Fisher information matrix, conditioned on $N_k > 0$, is given by*

$$\mathbf{I}_n(\alpha,\beta) = n\begin{pmatrix} \sum_{i=1}^{k} A_i(\tau) & \sum_{i=1}^{k} A_i(\tau)x_i \\ \sum_{i=1}^{k} A_i(\tau)x_i & \sum_{i=1}^{k} A_i(\tau)x_i^2 \end{pmatrix}, \tag{5.9}$$

where

$$\begin{aligned} A_i(\tau) &= \frac{E[N_i]}{n(1-[H_1(\tau)]^n)}\left[(1-V_i(\tau))F_i(\tau) + \frac{\tau}{\theta_i}(1-H_i(\tau))V_i(\tau)\right] \\ &= \frac{1}{1-[H_1(\tau)]^n}\left[\prod_{j=1}^{i-1} S_j(\tau)(1-\pi_j^*)\right] \\ &\quad \times\Big[(1-V_i(\tau))F_i(\tau) + \tau(1-H_i(\tau))V_i(\tau)\exp(\alpha+\beta x_i)\Big]. \end{aligned} \tag{5.10}$$

Before presenting numerical results, we make a remark on the asymptotic behavior of the distributional results obtained in this section. For this purpose, we first need to observe a simple property of recursive equation (5.2) as given below.

Theorem 5.3. *For $i = 1, 2, \dots, k$, we have $0 \le H_i(\tau) < 1$.*

From this property, it is apparent that $0 \le H_1(\tau) < 1$, and so it follows immediately from (5.3) that

$$\lim_{n\to\infty} Pr(N_k > 0) = 1 - \lim_{n\to\infty} [H_1(\tau)]^n = 1.$$

As expected, it reveals that the probability of a k-level step-stress test terminating at level x_k converges to 1 as the sample size n increases. Based on this observation, the following limits result:

$$\begin{aligned}
\lim_{n\to\infty} V_i(\tau) &= \frac{\lim_{n\to\infty} [H_1(\tau)]^{n-1}}{\prod_{j=1}^{i-1} [H_{j+1}(\tau)]^{\pi_j^*}} = 0,\\
\lim_{n\to\infty} E_c[n_i] &= E[n_i],\\
\lim_{n\to\infty} E_c[N_i] &= E[N_i]
\end{aligned}$$

for $i = 1, 2, \dots, k$. Consequently, from (5.10) in Theorem 5.2, we get

$$\lim_{n\to\infty} nA_i(\tau) = E[N_i]F_i(\tau) = E[n_i],$$

which is identical to $nA_i(\tau)$ in (3.5) defined earlier in Theorem 3.1. Thus, we observe that all the distributional results obtained in this section by conditioning on $N_k > 0$ ultimately converge to the unconditional results in Section 3 when the sample size n gets larger. Since the conditional information matrix of α and β in Theorem 5.2 eventually approaches the unconditional information matrix presented in Theorem 3.1, it is clear that the optimization results based on these information matrices should produce close results for large n. In other words, conditioning does not make much difference to the analysis when the initial sample size is large.

As done in Section 4, we can also define objective functions based on the conditional information matrix in (5.9) for determining optimal step duration using C-optimality, D-optimality, and A-optimality criteria. Unfortunately, the complexity of $A_i(\tau)$ in (5.10) makes it impossible to analytically prove the existence of the optimal τ even in the case of a simple step-stress testing. Nevertheless, the determination of optimal τ can be done numerically.

6. Numerical results

A numerical study was conducted in order to investigate the existence of the optimal stress change points and to evaluate them as a function of varying parameters ($viz.$, the sample size, MTTF, the number of stress levels, and the degree of censoring). For the purpose of illustration, we considered equi-spaced stress levels as $x_i = x_0 + id$ with the use-stress level $x_0 = 10$ and the stress increment $d = 5$. Under this setup, optimizing with respect to either the C-optimality or the D-optimality criterion is independent of the values of x_0 and d in the framework of Section 4. On the other hand, optimizing with respect to the A-optimality criterion is sensitive to the choice of x_0 and d. Moreover, optimization based on the conditional distribution results in Section 5 inherently depends on the sample size n under any optimality criterion since the sample size largely influences the probability of the test terminating at stress level x_k. We also chose the ordered MTTF as

$$\theta_{i+1} = \rho\theta_i, \qquad i = 1, 2, \dots, k-1, \qquad 0 < \rho < 1,$$

with selected choices of θ_1 and ρ. Under this setup, therefore, a decreasing geometric sequence of MTTF is simulated with an increasing arithmetic sequence of stress levels.

Tables 1 and 2 present the values of τ_C^*, τ_D^* and τ_A^* determined from the model in Section 3 for a feasible PC scheme. Rather than the specific values of the optimal stress durations, the tables are intended to provide a qualitative insight into the way the optimal choice changes as a function of the relevant parameters. To be able to compare the results with those from the large-sample results of Gouno, Sen and Balakrishnan (2004), the overall PC proportion was kept uniform *on average* for all stages. That is, we let $E[c_i] = n\pi_i$ or simply $\pi_i^* = \dfrac{n\pi_i}{E[N_i]S_i(\tau)}$ for $i = 1, 2, \ldots, k-1$, where π_i is constant for all i. Tables 3 and 4 present the values of the censoring proportion π_i^* at the optima achieved by the time points presented in Tables 1 and 2, respectively.

Surprisingly, Tables 1 and 2 are identical to the ones presented in Gouno, Sen and Balakrishnan (2004) even for the newly added A-optimality criterion. With the chosen parameters, the optimal stress change points under a large sample ($i.e.$, early termination of a test disallowed) coincide with the optimal points under the modification of censoring by proportion ($i.e.$, early termination of a test allowed) as long as the number of items progressively censored at each stress level is the same *on average*. Nevertheless, the advantages of the modified model in Section 3 are clear when a practitioner or a test designer has to deal with a small sample size, high censoring proportions, or quite a few stress levels. Such situations prohibit us from using the protocol based on a large sample because the search region $\mathcal{C}_\tau$ for the optimal τ may not be defined at all. However, the modified model suggested here does not impose any restrictions on $\mathcal{C}_\tau$ and consequently, the optimal stress change points can be searched for any combinations of the parameter values.

We now summarize the findings from Tables 1 and 2 below:

- It is observed that $\tau_C^* > \tau_D^* > \tau_A^*$ except for the simple step-stress case with $\rho = 0.5$. This order, however, is a consequence of the specific setting chosen and does not necessarily hold for general stress levels. For the example considered here, the differences among τ_C^*, τ_D^* and τ_A^* are more pronounced for the simple step-stress case and they reduce rapidly as the number of stress levels k increases. Also, for a given k and ρ, the ratios $\dfrac{\tau_C^*}{\tau_D^*}$ and $\dfrac{\tau_D^*}{\tau_A^*}$ seem to remain constant over varying ranges of θ_1, and they form a decreasing function of the overall PC proportion.

- The optimal values in Table 1 dominate the corresponding values in Table 2. Interestingly, for a fixed k and ρ, the percentage reduction in the optimal values in Table 2 with respect to the corresponding ones in Table 1 remains reasonably constant across the choices of θ_1. For $k = 2$, for instance, the ratio $\dfrac{\tau^*_{C,Table2}}{\tau^*_{C,Table1}}$ is roughly stable around 83.2% with ρ fixed. As ρ increases, $\dfrac{\tau^*_{C,Table2}}{\tau^*_{C,Table1}}$, $\dfrac{\tau^*_{D,Table2}}{\tau^*_{D,Table1}}$ and $\dfrac{\tau^*_{A,Table2}}{\tau^*_{A,Table1}}$ decrease slightly for a given k. The dependence on ρ, however, is less noticeable for smaller values of k. These ratios also decrease steadily with increasing k.

- The behavior of the optimal τ as a function of the MTTF values is also interesting. For fixed k and ρ, as θ_1 increases, τ_C^*, τ_D^* and τ_A^* increase in a manner such that the ratios τ_C^*/θ_1, τ_D^*/θ_1 and τ_A^*/θ_1 are constant across the values of θ_1. This translates to τ_C^*, τ_D^* and τ_A^* being fixed percentiles from the stage-1 distribution, irrespective of the value of θ_1. This feature prevails in both Tables 1 and 2.

- As the shrinkage amount ρ increases with θ_1 and k fixed, τ_C^*, τ_D^* and τ_A^* all increase in

Table 1: Optimal stress change points under the modification of $c_i = (N_i - n_i)\pi_i^*$ with the expected overall PC proportion being 10%

$\pi_i = 0.1$		$k=2$			$k=3$			$k=4$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	91.6	60.6	30.9	10.1	6.6	3.1	1.0	0.7	0.3
	$\rho = 0.3$	93.6	72.7	64.1	31.4	21.6	16.2	9.9	6.7	4.7
	$\rho = 0.5$	95.1	81.2	87.7	45.5	34.6	30.9	21.4	15.9	13.2
$\theta_1 = 300$	$\rho = 0.1$	274.9	181.7	92.8	30.4	19.9	9.2	2.9	2.1	1.0
	$\rho = 0.3$	280.7	218.0	192.4	94.2	64.7	48.7	29.6	20.0	14.1
	$\rho = 0.5$	285.4	243.5	263.0	136.6	103.8	92.8	64.1	47.7	39.5
$\theta_1 = 500$	$\rho = 0.1$	458.2	302.9	154.7	50.7	33.1	15.4	4.8	3.4	1.6
	$\rho = 0.3$	467.8	363.3	320.6	157.0	107.9	81.1	49.3	33.4	23.5
	$\rho = 0.5$	475.7	405.8	438.3	227.7	173.0	154.7	106.7	79.6	65.9

Table 2: Optimal stress change points under the modification of $c_i = (N_i - n_i)\pi_i^*$ with the expected overall PC proportion being 20%

$\pi_i = 0.2$		$k=2$			$k=3$			$k=4$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	76.3	52.3	29.5	7.2	5.1	2.8	0.6	0.5	0.3
	$\rho = 0.3$	77.9	63.1	59.1	20.8	16.3	13.9	5.0	4.2	3.6
	$\rho = 0.5$	78.4	69.3	79.0	30.0	25.3	25.4	10.8	9.4	9.4
$\theta_1 = 300$	$\rho = 0.1$	228.8	156.9	88.4	21.5	15.4	8.5	1.7	1.4	0.8
	$\rho = 0.3$	233.6	189.2	177.3	62.5	49.0	41.6	15.0	12.5	10.8
	$\rho = 0.5$	235.3	207.9	237.0	90.1	76.0	76.1	32.4	28.2	28.1
$\theta_1 = 500$	$\rho = 0.1$	381.3	261.5	147.4	35.9	25.7	14.2	2.9	2.3	1.4
	$\rho = 0.3$	389.4	315.3	295.5	104.2	81.7	69.4	25.0	20.8	17.9
	$\rho = 0.5$	392.2	346.6	395.0	150.2	126.6	126.8	54.0	47.1	46.8

such a way that the ratio of the increase is independent of the values of θ_1. Intuitively, this means that the more severe the successive stress levels are ($viz.$, smaller ρ), the more likely it is to observe failures in a short time interval. Hence, the choice of the optimal τ automatically forces the experiment to be terminated faster. The only exception is the simple step-stress case where it seems that ρ has very little effect in determining the optimal τ, especially τ_C^*.

- τ_C^*, τ_D^* and τ_A^* decrease quite rapidly as a function of k. In fact, both Tables 1 and 2 demonstrate that for $k = 4$ and small values of ρ, these optimal values are in the lower tail of the stage-1 life distribution. Consequently, it may frequently force to terminate the first stage of a life-test even before observing any failures. In that case, one practical strategy may be to continue the first-stage testing beyond τ_C^*, τ_D^* or τ_A^*.

As mentioned earlier, Tables 3 and 4 list the values of π_i^* required to produce each optimal τ in Tables 1 and 2. We see that these fixed PC proportions are at least the overall PC proportion specified, and for a fixed k and ρ, they form an increasing sequence ($i.e.$, $\pi_1^* < \pi_2^* < \cdots < \pi_{k-1}^*$) in order to keep the overall PC proportion uniform. In general, π_i^* is the highest for the C-optimality and the lowest for the A-optimality criterion. Of course, the higher the overall PC proportion is, the higher the fixed PC proportions are. One remark to make about these fixed PC proportions at the optima is that they do not depend on the

Table 3: Fixed PC proportions under the modification of $c_i = (N_i - n_i)\pi_i^*$ for the expected overall PC proportion at 10% with $\theta_1 = 100, 300, 500$

$\pi_i = 0.1$	$k=2$			$k=3$					
Optimality	C	D	A	C		D		A	
	π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*
$\rho = 0.1$	0.25	0.18	0.14	0.11	0.34	0.11	0.23	0.10	0.16
$\rho = 0.3$	0.25	0.21	0.19	0.14	0.45	0.12	0.29	0.12	0.23
$\rho = 0.5$	0.26	0.23	0.24	0.16	0.47	0.14	0.33	0.14	0.29

Table 4: Fixed PC proportions under the modification of $c_i = (N_i - n_i)\pi_i^*$ for the expected overall PC proportion at 20% with $\theta_1 = 100, 300, 500$

$\pi_i = 0.2$	$k=2$			$k=3$					
Optimality	C	D	A	C		D		A	
	π_1^*	π_1^*	π_1^*	π_1^*	π_2^*	π_1^*	π_2^*	π_1^*	π_2^*
$\rho = 0.1$	0.43	0.34	0.27	0.21	0.56	0.21	0.45	0.21	0.34
$\rho = 0.3$	0.44	0.38	0.36	0.25	0.65	0.24	0.53	0.23	0.47
$\rho = 0.5$	0.44	0.40	0.44	0.27	0.67	0.26	0.58	0.26	0.58

values of θ_1 but slightly increase with ρ. The dependence on ρ, however, is little for the first stage of the test.

Similarly, we also constructed the objective function for each optimality criterion, using the conditional distribution results established in Section 5. Tables 5 and 6 present the results of this numerical study for a simple step-stress case with varying sample sizes. Again, to be able to compare the results with those from Gouno, Sen and Balakrishnan (2004) as well as the values in Tables 1 and 2, the expected overall PC proportion was kept constant by setting $E_c[c_1] = n\pi_1$ or simply $\pi_1^* = \dfrac{n\pi_1}{n - E_c[n_1]} = \dfrac{\pi_1(1 - [F_1(\tau)]^n)}{S_1(\tau)}$. Tables 7 and 8 present these values of π_1^* at each optimal τ in Tables 5 and 6, respectively. From Tables 5 and 6, it is also noted that with the chosen parameters, the sample size required to produce the same optimal change points as in Tables 1 and 2 is at least 20. Intuitively, this means that the probability of a simple step-stress test terminating at the second stage is effectively 1 if the sample size is 20 or larger. Hence, we have numerically shown that the optimal τ conditioned on $N_k > 0$ converges to the unconditional one as the sample size increases.

Unfortunately, for small sample sizes, τ_A^* does not exist globally since the objective function $a(\tau)$ keeps increasing over the unrestricted range of τ. Thus, in the case of nonexistent τ_A^*, the choice of the optimal τ is completely up to the decision of a practitioner. In some cases, $a(\tau)$ exhibits a local maximum, and in order to capture this, we have imposed a constraint upon the search region for τ_A^* that the probability of observing a failure at the first stage should be at most 80%. That is, $F(\tau) \leq 0.8$ or equivalently $\tau \leq \theta_1 \log 5$. Findings from Tables 5 and 6 are similar to those from Tables 1 and 2. Additionally, we see that as n increases, τ_C^*, τ_D^* and τ_A^* all decrease but converge to their respective unconditional ones.

The value of π_1^* for each optimal τ in Tables 5 and 6 are tabulated in Tables 7 and 8, respectively. Again, these fixed PC proportions are greater than the specified overall PC proportion. We also observe that π_1^* is generally the highest for the C-optimality and the lowest for the A-optimality criterion under the chosen setting. Moreover, the fixed PC

Table 5: Optimal stress change points of the simple step-stress testing ($k = 2$) under the condition of $N_k > 0$ with the expected overall PC proportion being 10%

$\pi_1 = 0.1$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	119.6	71.2	(31.4)[a]	93.6	60.8	(30.9)	91.6	60.6	30.9
	$\rho = 0.3$	123.2	90.6	DNE[b]	95.7	73.3	(64.6)	93.6	72.7	64.1
	$\rho = 0.5$	130.5	113.6	DNE	97.7	82.5	(92.8)	95.1	81.2	87.7
$\theta_1 = 300$	$\rho = 0.1$	358.7	213.7	(94.2)	280.7	182.5	(92.8)	274.9	181.7	92.8
	$\rho = 0.3$	369.7	271.7	DNE	287.2	220.0	(193.7)	280.7	218.0	192.4
	$\rho = 0.5$	391.6	340.9	DNE	293.1	247.6	(278.4)	285.4	243.5	263.0
$\theta_1 = 500$	$\rho = 0.1$	597.9	356.2	(157.0)	467.9	304.1	(154.7)	458.2	302.9	154.7
	$\rho = 0.3$	616.1	452.9	DNE	478.7	366.7	(322.9)	467.8	363.3	320.6
	$\rho = 0.5$	652.6	568.1	DNE	488.5	412.7	(463.9)	475.7	405.8	438.3

Table 6: Optimal stress change points of the simple step-stress testing ($k = 2$) under the condition of $N_k > 0$ with the expected overall PC proportion being 20%

$\pi_1 = 0.2$		$n = 5$			$n = 10$			$n \geq 20$		
		τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*	τ_C^*	τ_D^*	τ_A^*
$\theta_1 = 100$	$\rho = 0.1$	87.1	56.9	(29.9)	76.8	52.4	29.5	76.3	52.3	29.5
	$\rho = 0.3$	89.8	71.2	DNE	78.5	63.3	59.4	77.9	63.1	59.1
	$\rho = 0.5$	91.9	81.8	DNE	79.1	69.8	81.3	78.4	69.3	79.0
$\theta_1 = 300$	$\rho = 0.1$	261.3	170.8	(89.6)	230.4	157.1	88.4	228.8	156.9	88.4
	$\rho = 0.3$	269.3	213.6	DNE	235.5	189.9	178.1	233.6	189.2	177.3
	$\rho = 0.5$	275.8	245.3	DNE	237.4	209.3	243.8	235.3	207.9	237.0
$\theta_1 = 500$	$\rho = 0.1$	435.5	284.6	(149.3)	384.0	261.9	147.4	381.3	261.5	147.4
	$\rho = 0.3$	448.8	356.0	DNE	392.5	316.5	296.8	389.4	315.3	295.5
	$\rho = 0.5$	459.6	408.8	DNE	395.7	348.8	406.3	392.2	346.6	395.0

[a] does not exist globally but locally exists under the constraint of $F(\tau) \leq 0.8$ or equivalently $\tau \leq \theta_1 \log 5$

[b] does not exist globally or locally

Table 7: Fixed PC proportions π_1^* of the simple step-stress testing ($k = 2$) under the condition of $N_k > 0$ for the expected overall PC proportion at 10% with $\theta_1 = 100, 300, 500$

$\pi_1 = 0.1$	$n = 5$			$n = 10$			$n \geq 20$		
Optimality	C	D	A	C	D	A	C	D	A
$\rho = 0.1$	0.28	0.20	(0.14)	0.25	0.18	(0.14)	0.25	0.18	0.14
$\rho = 0.3$	0.28	0.23	DNE	0.26	0.21	(0.19)	0.25	0.21	0.19
$\rho = 0.5$	0.29	0.27	DNE	0.26	0.23	(0.25)	0.26	0.23	0.24

Table 8: Fixed PC proportions π_1^* of the simple step-stress testing ($k = 2$) under the condition of $N_k > 0$ for the expected overall PC proportion at 20% with $\theta_1 = 100, 300, 500$

$\pi_1 = 0.2$	$n = 5$			$n = 10$			$n \geq 20$		
Optimality	C	D	A	C	D	A	C	D	A
$\rho = 0.1$	0.45	0.35	(0.27)	0.43	0.34	0.27	0.43	0.34	0.27
$\rho = 0.3$	0.46	0.39	DNE	0.44	0.38	0.36	0.44	0.38	0.36
$\rho = 0.5$	0.46	0.43	DNE	0.44	0.40	0.45	0.44	0.40	0.44

Table 9: Efficiency of the simple step-stress testing ($k = 2$) under the condition of $N_k > 0$ for the expected overall PC proportion at 10% & 20% with $\theta_1 = 100, 300, 500$

		$n = 5$			$n = 10$			$n \geq 20$		
Optimality		C	D	A	C	D	A	C	D	A
$\pi_1 = 0.1$	$\rho = 0.1$	0.92	1.04	(1.00)	1.00	1.00	(1.00)	1.00	1.00	1.00
	$\rho = 0.3$	0.91	1.09	DNE	0.99	1.00	(1.00)	1.00	1.00	1.00
	$\rho = 0.5$	0.91	1.16	DNE	1.00	1.03	(1.00)	1.00	1.00	1.00
$\pi_1 = 0.2$	$\rho = 0.1$	0.95	1.02	(1.00)	0.99	1.00	1.00	1.00	1.00	1.00
	$\rho = 0.3$	0.95	1.06	DNE	0.99	1.00	1.00	1.00	1.00	1.00
	$\rho = 0.5$	0.94	1.11	DNE	1.00	1.00	1.00	1.00	1.00	1.00

proportions get higher if the overall PC proportion increases, just like in Tables 3 and 4. What is interesting about these fixed PC proportions is that they are not dependent on θ_1 but exhibit a very slight increment with ρ. As expected, they form a decreasing convergent sequence to the unconditional π_1^* as n increases.

In an attempt to assess the efficiencies of the different approaches to the optimization problem and to contrast the results obtained here, pairwise ratios of the optima under each criterion were calculated based on the optimal stress change points determined by Gouno, Sen and Balakrishnan (2004) and by the results developed here. Since Tables 1 and 2 yield not only the identical stress change points but also exactly the same optima compared to the results in Gouno, Sen and Balakrishnan (2004), the efficiency between the modified (unconditional) model and the large sample model is not different with respect to the matched overall PC proportions. On the other hand, the efficiency of the conditional method relies upon the sample size n. Table 9 presents the ratios of the conditional optima to the unconditional ones for the simple step-stress case with varying sample sizes. Although these ratios are invariant across the values of θ_1, how they change with respect to other parameters is noticeable. With small n, large ρ and small π_1, we find that the efficiency of the conditional approach is a bit low for the C-optimality but is higher for the D-optimality. For both optimality criteria, however, the differences become negligible as n gets larger since the conditional optima eventually converge to the unconditional ones obtained from the modified model. Another interesting observation is that irrespective of the sample size, the constrained τ_A^* presented in Tables 5 and 6 attains the local optimum that is identical to the global maximum attained by τ_A^* from the modified (unconditional) model. Therefore, one can always choose to increase the efficiency of the conditional approach under the A-optimality criterion by selecting an arbitrary τ which bears a higher optimum than the one achieved by τ_A^* from the unconditional model. For boosting the efficiency, however, one must be prepared to take a drastic increase in the whole test duration, too.

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