Estimation and Evaluating of Aggregate Discounted Claims

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This paper focuses on important aspects of aggregate discounted claims. Estimation and Evaluation method will be used. The different distributions will be considered. Some numerical examples to illustrate the results will be given.

Key words: Aggregate Claims, Counting distribution, Claim distribution, Mixture Distribution, Losses.

1. Introduction

One important topic in the risk theory is probability of ruin. According to the notation of Bowers et al, the classical probability of ruin defined as follows. For any $t \geq 0$, let $U(t)$ denote the surplus at time t. Assume premiums are received continuously at a constant rate, c > 0, per unit time. Let S(t) denote the aggregate claims up to time t.

$$S(t) = X_1 + X_2 + ... + X_{N(t)}$$

Where X_1 denotes the amount of the first claim, X_2 be the amount of the second claim and so on. $N(t)$ denotes the number of claims produced by a portfolio of policies up to time t, and has the Poisson distribution with parameter λ. Assume $X_1, X_2, \ldots$ are independent claim random variables with the same probability distributed, and random variables N, X_1, $X_2, \ldots$ are mutually independent.

Let U(0) = u is the surplus at time 0, maybe as a result of past operations, then the surplus at time t will be

$$U(t) = u + ct - S(t)$$

In this model, we ignore factors such as, interest, expenses and dividends etc could affect surplus and only consider premiums and claims.

It is very clear that the surplus is increases function of time t with slope c except at those times when claims occur. Then the surplus drops by the amount of the claims. Therefore the surplus might become negative at certain times. When this first happens, we speak of ruin having occurred. We define

$$T = \min(t : U(t) < 0)$$

T is the first time the surplus below zero. If $T = \infty$, then surplus is always great than zero for all $t \geq 0$; i.e., ruin does not occur. When $T < \infty$, then ruin happens. Therefore find the probability of ruin and minimize it became a very important topic in the area of ruin theory. We define the probability of ruin is the function of the initial surplus which we use $\Psi(u)$ to denote. Then,

$$\Psi(u) = P(T < \infty)$$

Finding the probability of ruin is really challenging and the numerical evaluation is very hard. Therefore many researches considered the special cases. Chan and Zhang considered the probability distribution of claim is exponential or Geometric claims and found the recursive formula of probability ruin for those two cases.

Finding the probability of ruin, that is the probability of the first time the surplus drop below zero and finding the time of recovery or when the surplus attains a certain level are two important topics in ruin theory. Deng consider the special case the surplus attains a certain level for exponential, mixture exponential and geometric families.

In this paper, we are going to consider double exponential, mixture double exponential families.

2. Time of Recovery and Background

As what we mentioned before, we are most interesting after ruin occur, when will be time of recovery or surplus attains at certain level. Let b for any constant, and $b \geq u$, then define

$$T_b = \min(t \geq 0 : U(t) = b)$$

This is the first time when the surplus reaches level b.

In order to find the probability of recovery, we consider the more general case. We define

$$\Phi_n(u,b) = \Pr(U(1) \geq b, U(2) \geq b, \cdots U(n) \geq b \mid U(0) = u)$$

Therefore the probability of the surplus attain a certain level b is,

$$\Psi_n(u,b) = 1 - \Phi(u,b)$$

We can simplify by consider the discrete case and rewrite $\Phi(u,b)$ as follows. After the assumption of discrete, now $U(t)$ can rewrite as,

$$U(t) = u + cn - \sum_{i=1}^{n} X_i$$

Therefore,

$U(t) \geq b$ is equivalent $u + cn - \sum_{i=1}^{n} X_i \geq b > 0$ or $\sum_{i=1}^{n} X_i \leq u + cn - b$

Hence,

$$\Phi_n(u,b) = \Pr(X_1 \leq u + c - b, X_1 + X_2 \leq u + 2c - b, \cdots X_1 + X_2 + \cdots X_n \leq u + nc - b)$$

When we extend the definition of the stopping time T_b as follows,

$$T_b = \begin{cases} \min(t \geq 0 : U(t) = b), & b > u \\ \textit{the surplus dropped below b before even attained b}, & b < u \end{cases}$$

When b=0, the stopping time T_0 is the time of recovery; which is the first time the surplus reaches zero after ruin.

Deng considered the more general case T_b. Find the probability of surplus attains at certain level for exponential, mixture exponential and geometric families.

3. The probability of Recovery

The time of recovery is the special case of surplus attains level b when b=0. So we are going to consider the more general case for any positive number b.

3.1. Double Exponential Claim Distribution

By mathematical induction, we can show that the function $\Phi_n(u,b) = \Pr(U(1) \geq b, U(2) \geq b, \cdots U(n) \geq b \mid U(0) = u)$ satisfy the recursive relation as follows,

$$\Phi_n(u,b) = \int_{-\infty}^{u+c-b} \Phi_{n-1}(u+c-x,b) f(x)dx$$

Therefore the probability of the surplus attain a certain level b

$$\begin{aligned}\Psi_n(u,b) &= 1-\Phi(u,b)\\ &= 1- \int_{-\infty}^{u+c-b} \Phi_{n-1}(u+c-x,b) f(x)dx\\ &= 1- \int_{-\infty}^{u+c-b} 1-\Psi_{n-1}(u+c-x,b) f(x)dx\\ &= 1- \int_{-\infty}^{u+c-b} f(x)dx - \int_{-\infty}^{u+c-b} \Psi_{n-1}(u+c-x,b) f(x)dx\\ &= 1-F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_{n-1}(b+x,b) f(u+c-b-x)dx\end{aligned}$$

We denote,

$C_1(u,b) = u+c-b$,

$C_2(u,b) = u+2c-b = C_1(u,b)+c$

……

$C_n(u,b) = u+nc-b = C_{n-1}(u,b)+c$

Assume the claim family according to double exponential distribution.

$$f(x) = \frac{\alpha}{2} e^{-\alpha|x|}, \text{ where } -\infty < x < \infty$$

$$F(x) = \begin{cases} \frac{1}{2} e^{\alpha x} & x<0 \\ 1-\frac{1}{2} e^{-\alpha x} & x>0 \end{cases}$$

Then we have

1. For n=1

$$\Psi_1(y,b) = 1-\Phi_1(y,b) = 1-F(y+c-b) = \begin{cases} 1-\frac{1}{2} e^{\alpha(y+c-b)} & y+c-b<0 \\ \frac{1}{2} e^{-\alpha(y+c-b)} & y+c-b>0 \end{cases}$$

Since $C_1(u,b) = u+c-b > 0$, so $\Psi_1(u,b) = \frac{1}{2} e^{-\alpha C_1(u,b)}$

2. For n=2

$$\begin{aligned}\Psi_2(u,b) &= 1-F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_1(b+y,b) f(u+c-b-y)dy\\ &= \Psi_1(u,b) + \int_{-\infty}^{-c} \Psi_1(b+y,b) f(u+c-b-y)dy + \int_{-c}^{u+c-b} \Psi_1(b+y,b) f(u+c-b-y)dy\end{aligned}$$

Where

$$\int_{-\infty}^{-c} \Psi_1(b+y,b) f(u+c-b-y)dy = \int_{-\infty}^{-c} (1-\frac{1}{2} e^{\alpha(y+c)}) \frac{\alpha}{2} e^{-\alpha|u+c-b-y|} dy$$

$$= \int_{-\infty}^{-c} \frac{\alpha}{2} e^{\alpha[y-(u+c-b)]} dy - \int_{-\infty}^{-c} \frac{1}{2} e^{\alpha(y+c)}) \frac{\alpha}{2} e^{\alpha[y-(u+c-b)]} dy$$

$$= \frac{1}{2} e^{-\alpha C_2(u,b)} - \frac{1}{8} e^{-\alpha C_2(u,b)}$$

$$= \frac{3}{8} e^{-\alpha C_2(u,b)}$$

And

$$\int_{-c}^{u+c-b} \Psi_1(b+y,b) f(u+c-b-y) dy = \int_{-c}^{u+c-b} \frac{1}{2} e^{-\alpha(y+c)}) \frac{\alpha}{2} e^{-\alpha|u+c-b-y|} dy$$

$$= \int_{-c}^{u+c-b} \frac{1}{2} e^{-\alpha(y+c)}) \frac{\alpha}{2} e^{-\alpha[y-(u+c-b)]} dy$$

$$= \frac{\alpha}{4} C_2(u,b) e^{-\alpha C_2(u,b)}$$

Therefore,

$$\Psi_2(u,b) = \Psi_1(u,b) + \frac{3}{8} e^{-\alpha C_2(u,b)} + \frac{\alpha}{4} C_2(u,b) e^{-\alpha C_2(u,b)},$$

Or we can rewrite

$$\Psi_2(u,b) = \Psi_1(u,b) + \frac{1}{2^2} e^{-\alpha C_2(u,b)} \sum_{k=0}^{1} \frac{(\alpha C_2(u,b))^k}{k!} + \frac{1}{2^3} \frac{(\alpha C_2(u,b))^0}{0!} e^{-\alpha C_2(u,b)}$$

Since $C_2(u,b)$ is always positive. So for any y,

$$\Psi_2(y,b) = \Psi_1(y,b) + \frac{3}{8} e^{-\alpha|C_2(y,b)|} + \frac{\alpha}{4} | C_2(y,b) | e^{-\alpha|C_2(y,b)|}$$

3. For n=3

$$\Psi_3(u,b) = 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_2(b+y,b) f(u+c-b-y) dy$$

$$= 1 - F(u+c-b) +$$

$$+ \int_{-\infty}^{u+c-b} \left[\Psi_1(b+y,b) + \left(\frac{3}{8} + \frac{\alpha}{4} | C_2(b+y,b) | \right) e^{-\alpha|C_2(b+y,b)|} \right] f(u+c-b-y) dy$$

$$= 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_1(b+y,b) f(u+c-b-y) dy +$$

$$+ \int_{-\infty}^{u+c-b} \frac{3}{8} e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y) dy$$

$$+ \int_{-\infty}^{u+c-b} \frac{\alpha}{4} | C_2(b+y,b) | e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y) dy$$

$$= \Psi_2(y,b) + \int_{-\infty}^{u+c-b} \frac{3}{8} e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y) dy +$$

$$+ \int_{-\infty}^{u+c-b} \frac{\alpha}{4} | C_2(b+y,b) | e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y) dy$$

Where,

$$\int_{-\infty}^{u+c-b} \frac{3}{8} e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y) dy$$

$$= \int_{-\infty}^{-2c} \frac{3}{8} e^{\alpha(y+2c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy + \int_{-2c}^{u+c-b} \frac{3}{8} e^{-\alpha(y+2c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{3}{32} e^{-\alpha C_3(u,b)} + \frac{3}{16}\alpha C_3((u,b) e^{-\alpha C_3(u,b)}$$

$$\int_{-\infty}^{u+c-b} \frac{\alpha}{4} | C_2(b+y,b) | e^{-\alpha|C_2(b+y,b)|} f(u+c-b-y)dy$$

$$= \int_{-\infty}^{-2c} \frac{\alpha}{4} | C_2(b+y,b) | e^{-\alpha|C_2(b+y,b)|} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \int_{-2c}^{u+c-b} \frac{\alpha}{4} | C_2(b+y,b) | e^{-\alpha|C_2(b+y,b)|} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \int_{-\infty}^{-2c} -\frac{\alpha}{4}(y+2c) e^{\alpha(y+2c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \int_{-2c}^{u+c-b} \frac{\alpha}{4}(\mathrm{y}+2\mathrm{c}) e^{-\alpha(y+2c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{1}{32} e^{-\alpha C_3(u,b)} + \frac{\alpha^2}{16}\left(C_3(u,b)\right)^2 e^{-\alpha C_3(u,b)}$$

Therefore,

$$\Psi_3(u,b) = \Psi_2(y,b) + \frac{3}{32} e^{-\alpha C_3(u,b)} + \frac{3}{16}\alpha C_3((u,b) e^{-\alpha C_3(u,b)} + \frac{1}{32} e^{-\alpha C_3(u,b)} + \frac{\alpha^2}{16}\left(C_3(u,b)\right)^2 e^{-\alpha C_3(u,b)}$$

$$= \Psi_2(y,b) + \frac{1}{8} e^{-\alpha C_3(u,b)} + \frac{3}{16}\alpha C_3((u,b) e^{-\alpha C_3(u,b)} + \frac{\alpha^2}{16}\left(C_3(u,b)\right)^2 e^{-\alpha C_3(u,b)}$$

Or we can rewrite

$$\Psi_3(u,b) = \Psi_2(y,b) + \frac{1}{2^3} e^{-\alpha C_3(u,b)} \sum_{k=0}^{2} \frac{(\alpha C_3(u,b))^k}{k!} + \frac{1}{2^4} \frac{(\alpha C_3(u,b))^1}{1!} e^{-\alpha C_3(u,b)}$$

4. For n=4

$$\Psi_4(u,b) = 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_3(b+y,b) f(u+c-b-y)dy$$

$$= 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_2(b+y,b) f(u+c-b-y)dy$$

$$+ \int_{-\infty}^{u+c-b} \left(\frac{1}{8} + \frac{3}{16}\alpha | C_3(b+y,b) | + \frac{\alpha^2}{16} | C_3(b+y,b) |^2 \right) e^{-\alpha|C_3(b+y,b)|} f(u+c-b-y)dy$$

$$= \Psi_3(u,b) +$$

$$+ \int_{-\infty}^{u+c-b} \left(\frac{1}{8} + \frac{3}{16}\alpha | C_3(b+y,b) | + \frac{\alpha^2}{16} | C_3(b+y,b) |^2 \right) e^{-\alpha|C_3(b+y,b)|} f(u+c-b-y)dy$$

Where

$$\int_{-\infty}^{u+c-b} \frac{1}{8} e^{-\alpha|C_3(b+y,b)|} f(u+c-b-y)dy$$

$$= \int_{-\infty}^{-3c} \frac{1}{8} e^{\alpha(\mathrm{y}+3\mathrm{c})} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy + \int_{-3c}^{u+c-b} \frac{1}{8} e^{-\alpha(\mathrm{y}+3\mathrm{c})} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{1}{32} e^{-\alpha C_4(u,b)} + \frac{\alpha}{16} C_4(u,b) e^{-\alpha C_4(u,b)}$$

$$\int_{-\infty}^{u+c-b} \frac{3}{16}\alpha | C_3(b+y,b) | e^{-\alpha|C_3(b+y,b)|} f(u+c-b-y)dy$$

$$= \int_{-\infty}^{-3c} -\frac{3}{16}\alpha(y+3c) e^{\alpha(y+3c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \int_{-3\mathrm{c}}^{u+c-b} \frac{3}{16}\alpha(y+3c) e^{-\alpha(y+3c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{3}{2^7} e^{-\alpha C_4(u,b)} + \frac{3}{4^3}\alpha^2\left(C_4(u,b)\right)^2 e^{-\alpha C_4(u,b)}$$

$$\int_{-\infty}^{u+c-b} \frac{\alpha^2}{16} | C_3(b+y,b)|^2 \; e^{-\alpha|C_3(b+y,b)|} f(u+c-b-y)dy$$

$$= \int_{-\infty}^{-3c} \frac{1}{16}\alpha^2 (y+3c)^2 e^{\alpha(y+3c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \int_{-3c}^{u+c-b} \frac{1}{16}\alpha^2 (y+3c)^2 e^{-\alpha(y+3c)} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{1}{2^7} e^{-\alpha C_4(u,b)} + \frac{1}{2^4}\frac{\left(\alpha C_4(u,b)\right)^3}{3!} e^{-\alpha C_4(u,b)}$$

Therefore,

$$\Psi_4(u,b) = \Psi_3(u,b) + \frac{1}{32} e^{-\alpha C_4(u,b)} + \frac{\alpha}{16} C_4(u,b) e^{-\alpha C_4(u,b)} + \frac{3}{2^7} e^{-\alpha C_4(u,b)}$$

$$+ \frac{3}{4^3}\alpha^2\left(C_4(u,b)\right)^2 e^{-\alpha C_4(u,b)} + \frac{1}{2^7} e^{-\alpha C_4(u,b)} + \frac{1}{2^4}\frac{\left(\alpha C_4(u,b)\right)^3}{3!} e^{-\alpha C_4(u,b)}$$

$$= \Psi_3(u,b) + \frac{1}{2^4} e^{-\alpha C_4(u,b)} + \frac{\alpha}{2^4} C_4(u,b) e^{-\alpha C_4(u,b)}$$

$$+ \frac{3}{4^3}\alpha^2\left(C_4(u,b)\right)^2 e^{-\alpha C_4(u,b)} + \frac{1}{2^4}\frac{\left(\alpha C_4(u,b)\right)^3}{3!} e^{-\alpha C_4(u,b)}$$

Or we can rewrite

$$\Psi_4(u,b) = \Psi_3(y,b) + \frac{1}{2^4} e^{-\alpha C_4(u,b)} \sum_{k=0}^{3} \frac{(\alpha C_4(u,b))^k}{k!} + \frac{1}{2^5}\frac{(\alpha C_4(u,b))^2}{2!} e^{-\alpha C_4(u,b)}$$

In the above proof, we use the following lemma,

Lemma 3.1 $I(n) = \int_{-\infty}^{-k} (y+k)^{\mathrm{n}} e^{2\alpha y} dy$, then $I(n) = \frac{(-1)^n n!}{(2\alpha)^n}\frac{e^{-2\alpha k}}{2\alpha}$.

Proof: Use integral by part, we can easily find the recursive formula $I(n) = -\frac{n}{2\alpha} I(n-1)$,

and since $I(0) = \frac{1}{2\alpha} e^{-2\alpha k}$. This will proof the result.

Theorem 3.2: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ To be the first time when the surplus reaches level b. Then the probability of the surplus attain a certain level b of $\Psi_n(u,b)$ satisfy the following recursive formula when the claim distribution is the double exponential.

$$\Psi_{\mathrm{n}}(u,b) = \Psi_{\mathrm{n-1}}(y,b) + \frac{1}{2^{\mathrm{n}}} e^{-\alpha C_{\mathrm{n}}(u,b)} \sum_{k=0}^{\mathrm{n-1}} \frac{(\alpha C_{\mathrm{n}}(u,b))^k}{k!} + \frac{1}{2^{\mathrm{n+1}}}\frac{(\alpha C_{\mathrm{n}}(u,b))^{\mathrm{n-2}}}{(\mathrm{n-2})!} e^{-\alpha C_{\mathrm{n}}(u,b)}$$

Proof: We are going to use the mathematical induction method and lemma 3.1. We already show that for n=1, 2, 3 and 4, they are all true. Now we assume that for n it is true, we have,

$$\Psi_{\mathrm{n}}(u,b) = \Psi_{\mathrm{n-1}}(y,b) + \frac{1}{2^{\mathrm{n}}} e^{-\alpha C_{\mathrm{n}}(u,b)} \sum_{k=0}^{\mathrm{n-1}} \frac{(\alpha C_{\mathrm{n}}(u,b))^k}{k!} + \frac{1}{2^{\mathrm{n+1}}}\frac{(\alpha C_{\mathrm{n}}(u,b))^{\mathrm{n-2}}}{(\mathrm{n-2})!} e^{-\alpha C_{\mathrm{n}}(u,b)}$$

Now, for n+1, we have

$$\Psi_{n+1}(u,b)=1-F(u+c-b)+\int_{-\infty}^{u+c-b}\Psi_n(b+y,b)f(u+c-b-y)dy$$

By using the result is true for n, we have

$$\Psi_{n+1}(u,b)=1-F(u+c-b)+\int_{-\infty}^{u+c-b}\Psi_{n-1}(b+y,b)f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\frac{1}{2^{n}}e^{-\alpha|C_n(b+y,b)|}\sum_{k=0}^{n-1}\frac{(\alpha C_n\,|(b+y,b)|)^k}{k!}f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\frac{1}{2^{n+1}}e^{-\alpha|C_n(b+y,b)|}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^{n-2}}{(n-2)!}f(u+c-b-y)dy$$

$$=\Psi_n(u,b)+$$

$$+\int_{-\infty}^{u+c-b}\frac{1}{2^{n}}e^{-\alpha|C_n(b+y,b)|}\sum_{k=0}^{n-1}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^k}{k!}f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\frac{1}{2^{n+1}}e^{-\alpha|C_n(b+y,b)|}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^{n-2}}{(n-2)!}f(u+c-b-y)dy$$

Where

$$\int_{-\infty}^{u+c-b}\frac{1}{2^{n}}e^{-\alpha|C_n(b+y,b)|}\sum_{k=0}^{n-1}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^k}{k!}f(u+c-b-y)dy$$

$$=\int_{-\infty}^{-nc}\frac{1}{2^{n}}e^{-\alpha|C_n(b+y,b)|}\sum_{k=0}^{n-1}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^k}{k!}\frac{\alpha}{2}e^{\alpha(y-(u+c-b))}dy+$$

$$+\int_{-nc}^{u+c-b}\frac{1}{2^{n}}e^{-\alpha|C_n(b+y,b)|}\sum_{k=0}^{n-1}\frac{(\alpha\,|\,C_n(b+y,b)\,|)^k}{k!}\frac{\alpha}{2}e^{\alpha(y-(u+c-b))}dy$$

$$=\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}\int_{-\infty}^{-nc}(-1)^k(y+nc)^k e^{\alpha(y+nc)}e^{\alpha(y-(u+c-b))}dy+$$

$$+\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}\int_{-nc}^{u+c-b}(y+nc)^k e^{-\alpha(y+nc)}e^{\alpha(y-(u+c-b))}dy$$

$$=\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}(-1)^k e^{\alpha(nc-(u+c-b))}\int_{-\infty}^{-nc}(y+nc)^k e^{2\alpha y}dy+$$

$$+\frac{1}{2^{n+1}}e^{-\alpha C_{n+1}(u,b)}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}\int_{-nc}^{u+c-b}(y+nc)^k dy$$

$$=\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}(-1)^k e^{\alpha(nc-(u+c-b))}I(k)+$$

$$+\frac{1}{2^{n+1}}e^{-\alpha C_{n+1}(u,b)}\sum_{k=0}^{n-1}\frac{(\alpha C_{n+1}(u,b))^{k+1}}{(k+1)!}$$

$$=\frac{1}{2^{n+1}}\sum_{k=0}^{n-1}\frac{\alpha^{k+1}}{k!}(-1)^k e^{\alpha(nc-(u+c-b))}\frac{(-1)^k k!}{(2\alpha)^k}\frac{e^{-2\alpha nc}}{2\alpha}+$$

$$+\frac{1}{2^{n+1}}e^{-\alpha C_{n+1}(u,b)}\sum_{k=1}^{n}\frac{(\alpha C_{n+1}(u,b))^{k}}{k!}$$

$$=\frac{1}{2^{n+1}}e^{-\alpha C_{n+1}(u,b)}\sum_{k=0}^{n-1}\frac{1}{2^{k+1}}+\frac{1}{2^{n+1}}e^{-\alpha C_{n+1}(u,b)}\sum_{k=1}^{n}\frac{(\alpha C_{n+1}(u,b))^{k}}{k!}$$

$$\int_{-\infty}^{u+c-b} \frac{1}{2^{n+1}} e^{-\alpha|C_n(b+y,b)|} \frac{(\alpha\,|\,C_n(b+y,b)\,|)^{n-2}}{(n-2)!} f(u+c-b-y)dy$$

$$= \int_{-\infty}^{-nc} \frac{1}{2^{n+1}} e^{-\alpha|C_n(b+y,b)|} \frac{(\alpha C_n\,|(b+y,b)\,|)^{n-2}}{(n-2)!} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \int_{-nc}^{u+c-b} \frac{1}{2^{n+1}} e^{-\alpha|C_n(b+y,b)|} \frac{(\alpha C_n\,|(b+y,b)\,|)^{n-2}}{(n-2)!} \frac{\alpha}{2} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} (-1)^{n-2} \int_{-\infty}^{-nc} (y+nc)^{n-2} e^{\alpha(y+nc)} e^{\alpha(y-(u+c-b))} dy +$$

$$+ \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} \int_{-nc}^{u+c-b} (y+nc)^{n-2} e^{-\alpha(y+nc)} e^{\alpha(y-(u+c-b))} dy$$

$$= \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} (-1)^{n-2} e^{\alpha(nc-(u+c-b))} \int_{-\infty}^{-nc} (y+nc)^{n-2} e^{2\alpha y} dy +$$

$$+ \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} e^{-\alpha C_{n+1}(u,b)} \int_{-nc}^{u+c-b} (y+nc)^{n-2} dy$$

$$= \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} (-1)^{n-2} e^{\alpha(nc-(u+c-b))} I(n-2) +$$

$$+ \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

$$= \frac{1}{2^{n+2}} \frac{\alpha^{n-1}}{(n-2)!} (-1)^{n-2} e^{\alpha(nc-(u+c-b))} \frac{(-1)^{n-2}(n-2)!}{(2\alpha)^{n-2}} \frac{e^{-2\alpha nc}}{2\alpha} +$$

$$+ \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

$$= \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \frac{1}{2^{n}} + \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

Therefore,

$$\Psi_{n+1}(u,b) = \Psi_n(u,b) +$$

$$+ \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} + \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \sum_{k=1}^{n} \frac{(\alpha C_{n+1}(u,b))^k}{k!} +$$

$$+ \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \frac{1}{2^{n}} + \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

$$= \Psi_n(u,b) + \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \left(\sum_{k=0}^{n-1} \frac{1}{2^{k+1}} + \frac{1}{2^{n}} \right) + \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \sum_{k=1}^{n} \frac{(\alpha C_{n+1}(u,b))^k}{k!} +$$

$$+ \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

$$= \Psi_n(y,b) + \frac{1}{2^{n+1}} e^{-\alpha C_{n+1}(u,b)} \sum_{k=0}^{n} \frac{(\alpha C_{n+1}(u,b))^k}{k!} + \frac{1}{2^{n+2}} \frac{(\alpha C_{n+1}(u,b))^{n-1}}{(n-1)!} e^{-\alpha C_{n+1}(u,b)}$$

This proved the theorem.

Next we define $P_n(u,b) = \Psi_n(u,b) - \Psi_{n-1}(u,b)$ when $n \geq 2$. The notation $P_n(u,b)$ is the probability that surplus attain a certain level b at instant time n. Therefore,

$$\Psi_n(u,b) - \Psi_{n-1}(u,b) = \Pr(X_1 > u+c-b, X_1+X_2 > u+2c-b, \cdots X_1+X_2+\cdots X_n > u+nc-b) -$$

$$- \Pr(X_1 > u+c-b, X_1+X_2 > u+2c-b, \cdots X_1+X_2+\cdots X_{n-1} > u+(n-1)c-b)$$

Hence, we are going to have the following Corollary.

Corollary 3.3: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ to be the first time when the surplus reaches level b. Then the probability that surplus attain a certain level b at instant time n of $P_n(u,b)$ satisfy the following recursive formula when the claim distribution is double exponential.

$$P_n(u,b) = \frac{1}{2^{\mathrm{n}}} e^{-\alpha C_{\mathrm{n}}(u,b)} \sum_{k=0}^{\mathrm{n-1}} \frac{(\alpha C_{\mathrm{n}}(u,b))^k}{k!} + \frac{1}{2^{\mathrm{n+1}}} \frac{(\alpha C_{\mathrm{n}}(u,b))^{\mathrm{n-2}}}{(\mathrm{n-2})!} e^{-\alpha C_{\mathrm{n}}(u,b)}$$

Proof: It can be directly proved by the definition and Theorem 3.2.

Corollary 3.4: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ to be the first time when the surplus reaches level b. Then the probability of the surplus attain a certain level b of $\Psi_n(u,b)$ satisfy the following formula when the claim distribution is double exponential.

$$\Psi_n(u,b) = \sum_{j=1}^{n} \sum_{k=0}^{\mathrm{j-1}} \frac{1}{2^{\mathrm{j}}} e^{-\alpha C_{\mathrm{j}}(u,b)} \frac{(\alpha C_{\mathrm{j}}(u,b))^k}{k!} + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j+1}}} \frac{(\alpha C_{\mathrm{j}}(u,b))^{\mathrm{j-2}}}{(\mathrm{j-2})!} e^{-\alpha C_{\mathrm{j}}(u,b)}$$

Proof: By Theorem 3.2 and Corollary 3.3, we have

$$\Psi_n(u,b) = \Psi_1(u,b) + \sum_{k=2}^{n} P_k(u,b)$$

$$= \Psi_1(u,b) + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j}}} e^{-\alpha C_{\mathrm{j}}(u,b)} \sum_{k=0}^{\mathrm{j-1}} \frac{(\alpha C_{\mathrm{j}}(u,b))^k}{k!} + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j+1}}} \frac{(\alpha C_{\mathrm{j}}(u,b))^{\mathrm{j-2}}}{(\mathrm{j-2})!} e^{-\alpha C_{\mathrm{j}}(u,b)}$$

$$= \frac{1}{2} e^{-\alpha C_1(u,b)} + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j}}} e^{-\alpha C_{\mathrm{j}}(u,b)} \sum_{k=0}^{\mathrm{j-1}} \frac{(\alpha C_{\mathrm{j}}(u,b))^k}{k!} + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j+1}}} \frac{(\alpha C_{\mathrm{j}}(u,b))^{\mathrm{j-2}}}{(\mathrm{j-2})!} e^{-\alpha C_{\mathrm{j}}(u,b)}$$

$$= \sum_{j=1}^{n} \sum_{k=0}^{\mathrm{j-1}} \frac{1}{2^{\mathrm{j}}} e^{-\alpha C_{\mathrm{j}}(u,b)} \frac{(\alpha C_{\mathrm{j}}(u,b))^k}{k!} + \sum_{j=2}^{n} \frac{1}{2^{\mathrm{j+1}}} \frac{(\alpha C_{\mathrm{j}}(u,b))^{\mathrm{j-2}}}{(\mathrm{j-2})!} e^{-\alpha C_{\mathrm{j}}(u,b)}$$

This proved the corollary.

3.2 Mixture of Double Exponential Claim Distribution:

At first, we consider the claim family according to mixture of double exponential distribution.

$$f(x) = \sum_{i=1}^{M} A_i \frac{\alpha_i}{2} e^{-\alpha_i |x|}, \text{ where } -\infty < x < \infty$$

$$F(x) = \begin{cases} \sum_{i=1}^{M} \frac{A_i}{2} e^{\alpha_i x} & x < 0 \\ 1 - \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i x} & x > 0 \end{cases}$$

Where $\sum_{i=1}^{M} A_i = 1$, A_i can be negative as long as the $f(x)$ is nonnegative. We also A_i satisfy the following conditions $\sum_{j \neq k} A_j^n A_k = 0$ for all n.

By the results above,

$$\Phi_n(u,b) = \int_{-\infty}^{u+c-b} \Phi_{n-1}(u+c-x,b)f(x)dx$$

the probability of the surplus attain a certain level b

$$\Psi_n(u,b) = 1-\Phi(u,b) \; = 1-F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_{n-1}(b+x,b)f(u+c-b-x)dx$$

and the notation of recursive relation, we have

$C_1(u,b) = u+c-b$ ， $C_n(u,b) = u+nc-b = C_{n-1}(u,b)+c$ hold for all n.

Assume the mixture of Double Exponential claim distribution, we have

1. For n=1

$$\Psi_1(y,b) = \begin{cases} 1 - \sum_{i=1}^{M} \frac{A_i}{2} e^{\alpha_i (y+c-b)} & y+c-b<0 \\ \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i (y+c-b)} & y+c-b>0 \end{cases}$$

When $y = u$, $u+c-b>0$ $\Psi_1(u,b) = 1-F(u+c-b) = \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i C_1(u,b)}$

2. For n=2

$$\Psi_2(u,b) = 1-F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_1(b+y,b)f(u+c-b-y)dy$$

$$= \Psi_1(u,b) + \int_{-\infty}^{-c} (1 - \sum_{i=1}^{M} \frac{A_i}{2} e^{\alpha_i (y+c)}) \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{-\alpha_j |u+c-b-y|} dy +$$

$$+ \int_{-c}^{u+c-b} \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i (y+c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{-\alpha_j |u+c-b-y|} dy$$

where

$$\int_{-\infty}^{-c} (1 - \sum_{i=1}^{M} \frac{A_i}{2} e^{\alpha_i (y+c)}) \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{-\alpha_j |u+c-b-y|} dy =$$

$$= \int_{-\infty}^{-c} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy - \int_{-\infty}^{-c} \sum_{i=1}^{M} \frac{A_i}{2} e^{\alpha_i (y+c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

$$= \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{-\alpha_j (u+c-b))} \int_{-\infty}^{-c} e^{\alpha_j y} dy - \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} e^{\alpha_j \cdot (u-b))} \int_{-\infty}^{-c} e^{2\alpha_i y} dy$$

$$= \sum_{j=1}^{M} \frac{A_j}{2} e^{-\alpha_j C_2(u,b)} - \sum_{j=1}^{M} \frac{A_j^2}{2^3} e^{-\alpha_j C_2(u,b)}$$

$$\int_{-c}^{u+c-b} \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i (y+c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{-\alpha_j |u+c-b-y|} dy =$$

$$= \int_{-c}^{u+c-b} \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i (y+c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

$$= \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} e^{-\alpha_j C_2(u,b)} \int_{-c}^{u+c-b} dy = \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} C_2(u,b) e^{-\alpha_j C_2(u,b)}$$

So,

$$\Psi_2(u,b) = \Psi_1(u,b) + \sum_{j=1}^{M} \frac{A_j}{2} e^{-\alpha_j C_2(u,b)} - \sum_{j=1}^{M} \frac{A_j^2}{2^3} e^{-\alpha_j C_2(u,b)} + \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} C_2(u,b) e^{-\alpha_j C_2(u,b)}$$

$$= \Psi_1(u,b) + \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j C_2(u,b)} + \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} C_2(u,b) e^{-\alpha_j C_2(u,b)}$$

Or we can rewrite

$$\Psi_2(u,b) = \Psi_1(u,b) + \frac{1}{2^2} \sum_{k=0}^{0} \sum_{j=1}^{M} A_j \frac{(\alpha_j C_2(u,b))^k}{k!} e^{-\alpha_j C_2(u,b)} +$$

$$+ \sum_{j=1}^{M} \frac{A_j(2 - A_j)}{2^3} e^{-\alpha_j C_2(u,b)} + \sum_{j=1}^{M} \frac{A_j^2}{2^2} \frac{(\alpha_j C_2(u,b))}{1!} e^{-\alpha_j C_2(u,b)}$$

Since $C_2(u,b)$ is always positive. So for any y,

$$\Psi_2(y,b) = \Psi_1(y,b) + \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j |C_2(y,b)|} + \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} |C_2(y,b)| e^{-\alpha_j |C_2(y,b)|}$$

3. For n=3

$$\Psi_3(u,b) = 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_2(b+y,b) f(u+c-b-y) dy$$

$$= 1 - F(u+c-b) + \int_{-\infty}^{u+c-b} \Psi_1(b+y,b) f(u+c-b-y) dy$$

$$+ \int_{-\infty}^{u+c-b} \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j |C_2(b+y,b)|} f(u+c-b-y) dy$$

$$+ \int_{-\infty}^{u+c-b} \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} |C_2(b+y,b)| e^{-\alpha_j |C_2(b+y,b)|} f(u+c-b-y) dy$$

$$= \Psi_2(u,b) + \int_{-\infty}^{u+c-b} \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j |C_2(b+y,b)|} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

$$+ \int_{-\infty}^{u+c-b} \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} |C_2(b+y,b)| e^{-\alpha_j |C_2(b+y,b)|} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

Where,

$$\int_{-\infty}^{u+c-b} \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j |C_2(b+y,b)|} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy =$$

$$= \int_{-\infty}^{-2c} \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{\alpha_j (y+2c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy +$$

$$+ \int_{-2c}^{u+c-b} \sum_{j=1}^{M} \frac{A_j(4 - A_j)}{2^3} e^{-\alpha_j (y+2c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

$$= \sum_{j=1}^{M} \frac{A_j^2(4 - A_j)\alpha_j}{2^4} e^{\alpha_j (2c-(u+c-b))} \int_{-\infty}^{-2c} e^{2\alpha_j y} dy + \sum_{j=1}^{M} \frac{A_j^2(4 - A_j)\alpha_j}{2^4} e^{-\alpha_j C_3(u,b)} \int_{-2c}^{u+c-b} dy$$

$$= \sum_{j=1}^{M} \frac{A_j^2(4 - A_j)}{2^5} e^{-\alpha_j C_3(u,b)} + \sum_{j=1}^{M} \frac{A_j^2(4 - A_j)\alpha_j}{2^4} C_3(u,b) e^{-\alpha_j C_3(u,b)}$$

$$\int_{-\infty}^{u+c-b} \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} |C_2(b+y,b)| e^{-\alpha_j |C_2(b+y,b)|} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy$$

$$= \int_{-\infty}^{-2c} \sum_{j=1}^{M} A_j^2 \frac{\alpha_j}{2^2} (-(y+2c)) e^{\alpha_j (y+2c)} \sum_{j=1}^{M} A_j \frac{\alpha_j}{2} e^{\alpha_j (y-(u+c-b))} dy +$$

$$+\int_{-2c}^{u+c-b}\sum_{j=1}^{M}A_j^2\frac{\alpha_j}{2^2}(\mathrm{y}+2\mathrm{c})\mathrm{e}^{-\alpha_j(\mathrm{y}+2\mathrm{c})}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(\mathrm{y}-(u+c-b))}dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}e^{\alpha_j(-\mathrm{u+c+b})}\int_{-\infty}^{-2c}(-(\mathrm{y}+2\mathrm{c}))e^{2\alpha_j\mathrm{y}}dy+\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}\int_{-2c}^{u+c-b}(\mathrm{y}+2\mathrm{c})dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}e^{\alpha_j(-\mathrm{u+c+b})}\frac{1}{(2\alpha)^2}e^{-4\alpha_j\mathrm{c}}+\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}\frac{\left(C_3(u,b)\right)^2}{2!}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

$$=\sum_{j=1}^{M}\frac{A_j^3}{2^5}e^{-\alpha_j\mathrm{C}_3(u,b)}+\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}\frac{\left(C_3(u,b)\right)^2}{2!}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

Therefore,

$$\Psi_3(u,b)=\Psi_2(u,b)+\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^5}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}+\sum_{j=1}^{M}\frac{A_j^2(4-A_j)\alpha_j}{2^4}C_3(u,b)e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

$$+\sum_{j=1}^{M}\frac{A_j^3}{2^5}e^{-\alpha_j\mathrm{C}_3(u,b)}+\sum_{j=1}^{M}\frac{A_j^3\alpha_j^2}{2^3}\frac{\left(C_3(u,b)\right)^2}{2!}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

$$=\Psi_2(u,b)+\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}+\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_jC_3(u,b)e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

$$+\sum_{j=1}^{M}\frac{A_j^3}{2^3}\frac{\left(\alpha_jC_3(u,b)\right)^2}{2!}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

Or we can rewrite,

$$\Psi_3(u,b)=\Psi_2(u,b)+\frac{1}{2^3}\sum_{k=0}^{1}\sum_{j=1}^{M}A_j^2\frac{(\alpha_j\mathrm{C}_3(\mathrm{u,b}))^{\mathrm{k}}}{k!}e^{-\alpha_j\mathrm{C}_3(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^2(2-A_j)}{2^4}(\alpha_jC_3(u,b))e^{-\alpha_j\mathrm{C}_3(u,b)}+\sum_{j=1}^{M}\frac{A_j^3}{2^3}\frac{(\alpha_jC_3(u,b))^2}{2!}e^{-\alpha_j\mathrm{C}_3(\mathrm{u,b})}$$

4. For n=4

$$\Psi_4(u,b)=1-F(u+c-b)+\int_{-\infty}^{u+c-b}\Psi_3(b+y,b)f(u+c-b-y)dy$$

$$=1-F(u+c-b)+\int_{-\infty}^{u+c-b}\Psi_2(b+y,b)f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{-\alpha_j|\mathrm{C}_3(b+y,b)|}f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_j\,|\,C_3(b+y,b)\,|\,e^{-\alpha_j|\mathrm{C}_3(b+y,b)|}f(u+c-b-y)dy$$

$$+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^3}{2^3}\alpha_j^2\frac{|\,C_2(b+y,b)\,|^2}{2!}e^{-\alpha_j|\mathrm{C}_3(\mathrm{b+y,b})|}f(u+c-b-y)dy$$

$$=\Psi_3(u,b)+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{-\alpha_j|\mathrm{C}_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(\mathrm{y}-(u+c-b))}dy$$

$$+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_j\,|\,C_3(b+y,b)\,|\,e^{-\alpha_j|\mathrm{C}_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(\mathrm{y}-(u+c-b))}dy$$

$$+\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^3}{2^3}\alpha_j^2\frac{|C_2(b+y,b)|^2}{2!}e^{-\alpha_j|C_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy$$

Where,

$$\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{-\alpha_j|C_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy=$$

$$=\int_{-\infty}^{-3c}\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy+\int_{-3c}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2}{2^3}e^{-\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3}{2^4}\alpha_j e^{\alpha_j(3c-(u+c-b))}\int_{-\infty}^{-3c}e^{2\alpha_j y}dy+\sum_{j=1}^{M}\frac{A_j^3}{2^4}\alpha_j\int_{-3c}^{u+c-b}e^{-\alpha_j C_4(u,b)}dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3}{2^5}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^3}{2^4}\alpha_j C_4(u,b)e^{-\alpha_j C_4(u,b)}$$

$$\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_j|C_3(b+y,b)|e^{-\alpha_j|C_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy=$$

$$=\int_{-\infty}^{-3c}\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_j(-(y+3c))e^{\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy+$$

$$+\int_{-3c}^{u+c-v}\sum_{j=1}^{M}\frac{A_j^2(4-A_j)}{2^4}\alpha_j(y+3c)e^{-\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\alpha_j^2 e^{\alpha_j(3c-(u+c-b))}\int_{-\infty}^{-3c}(-(y+3c))e^{2\alpha_j y}dy+$$

$$+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\alpha_j^2 e^{-\alpha_j C_4(u,b)}\int_{-3c}^{u+c-b}(y+3c)dy$$

$$=\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\alpha_j^2 e^{\alpha_j(3c-(u+c-b))}\frac{e^{-6\alpha_j c}}{(2\alpha_j)^2}+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\frac{(\alpha C_4(u,b))^2}{2!}e^{-\alpha_j C_4(u,b)}$$

$$=\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^7}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\frac{(\alpha C_4(u,b))^2}{2!}e^{-\alpha_j C_4(u,b)}$$

$$\int_{-\infty}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^3}{2^3}\alpha_j^2\frac{|C_2(b+y,b)|^2}{2!}e^{-\alpha_j|C_3(b+y,b)|}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy=$$

$$=\int_{-\infty}^{-3c}\sum_{j=1}^{M}\frac{A_j^3}{2^3}\alpha_j^2\frac{(y+3c)^2}{2!}e^{\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy+$$

$$+\int_{-3c}^{u+c-b}\sum_{j=1}^{M}\frac{A_j^3}{2^3}\alpha_j^2\frac{(y+3c)^2}{2!}e^{-\alpha_j(y+3c)}\sum_{j=1}^{M}A_j\frac{\alpha_j}{2}e^{\alpha_j(y-(u+c-b))}dy$$

$$=\sum_{j=1}^{M}\frac{A_j^4}{2^4}\alpha_j^3\frac{1}{2!}e^{\alpha_j(3c-(u+c-b))}\int_{-\infty}^{-3c}(y+3c)^2e^{2\alpha_j y}dy+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\alpha_j^3\frac{1}{2!}e^{-\alpha_j C_4(u,b)}\int_{-3c}^{u+c-b}(y+3c)^2dy$$

$$=\sum_{j=1}^{M}\frac{A_j^4}{2^4}\alpha_j^3\frac{1}{2!}e^{\alpha_j(3c-(u+c-b))}\frac{2!}{(2\alpha_j)^2}\frac{e^{-2\alpha_j(3c)}}{2\alpha_j}+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\alpha_j^3\frac{1}{2!}e^{-\alpha_j C_4(u,b)}\frac{(C_4(u,b))^3}{3}$$

$$=\sum_{j=1}^{M}\frac{A_j^4}{2^7}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\frac{(\alpha_j C_4(u,b))^3}{3!}e^{-\alpha_j C_4(u,b)}$$

Therefore,

$$\Psi_4(u,b)=\Psi_3(u,b)+\sum_{j=1}^{M}\frac{A_j^3}{2^5}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^3}{2^4}\alpha_j C_4(u,b)e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^7}e^{-\alpha_j C_4(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\frac{(\alpha C_4(u,b))^2}{2!}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^4}{2^7}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\frac{(\alpha_j C_4(u,b))^3}{3!}e^{-\alpha_j C_4(u,b)}$$

$$=\Psi_3(u,b)+\sum_{j=1}^{M}\frac{A_j^3}{2^4}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^3}{2^4}\alpha_j C_4(u,b)e^{-\alpha_j C_4(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^3(4-A_j)}{2^5}\frac{(\alpha C_4(u,b))^2}{2!}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\frac{(\alpha_j C_4(u,b))^3}{3!}e^{-\alpha_j C_4(u,b)}$$

Or we can rewrite,

$$\Psi_4(u,b)=\Psi_3(u,b)+\frac{1}{2^4}\sum_{k=0}^{2}\sum_{j=1}^{M}A_j^3\frac{(\alpha_j C_4(u,b))^k}{k!}e^{-\alpha_j C_4(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^3(2-A_j)}{2^5}\frac{(\alpha_j C_4(u,b))^2}{2!}e^{-\alpha_j C_4(u,b)}+\sum_{j=1}^{M}\frac{A_j^4}{2^4}\frac{(\alpha_j C_4(u,b))^3}{3!}e^{-\alpha_j C_4(u,b)}$$

Theorem 3.5: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ To be the first time when the surplus reaches level b. Then the probability of the surplus attain a certain level b of $\Psi_n(u,b)$ satisfy the following recursive formula when the claim distribution is the mixture double exponential.

$$\Psi_n(u,b)=\Psi_{n-1}(u,b)+\frac{1}{2^n}\sum_{k=0}^{n-2}\sum_{j=1}^{M}A_j^{n-1}\frac{(\alpha_j C_n(u,b))^k}{k!}e^{-\alpha_j C_n(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^{n-1}(2-A_j)}{2^{n+1}}\frac{(\alpha_j C_n(u,b))^{n-2}}{(n-2)!}e^{-\alpha_j C_n(u,b)}+\sum_{j=1}^{M}\frac{A_j^n}{2^n}\frac{(\alpha_j C_n(u,b))^{n-1}}{(n-1)!}e^{-\alpha_j C_n(u,b)}$$

Proof: We already show when n=2, 3, and 4 are true. The theorem can be proved by mathematical induction.

By the similarity, we define $P_n(u,b) = \Psi_n(u,b) - \Psi_{n-1}(u,b)$, when $n \geq 2$. The notation $P_n(u,b)$ is the probability that surplus attain a certain level b at instant time n. Therefore,

$$\Psi_n(u,b)-\Psi_{n-1}(u,b)=\Pr(X_1>u+c-b, X_1+X_2>u+2c-b, \cdots X_1+X_2+\cdots X_n>u+nc-b)-$$
$$-\Pr(X_1>u+c-b, X_1+X_2>u+2c-b, \cdots X_1+X_2+\cdots X_{n-1}>u+(n-1)c-b)$$

Hence, we are going to have the following Corollary.

Corollary 3.6: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ to be the first time when the surplus reaches level b. Then the probability that surplus attain a certain level b at instant time n of $P_n(u,b)$ satisfy the following recursive formula when the claim distribution is the mixture double exponential.

$$P_n(u,b)=\frac{1}{2^n}\sum_{k=0}^{n-2}\sum_{j=1}^{M}A_j^{n-1}\frac{(\alpha_j C_n(u,b))^k}{k!}e^{-\alpha_j C_n(u,b)}+\sum_{j=1}^{M}\frac{A_j^{n-1}(2-A_j)}{2^{n+1}}\frac{(\alpha_j C_n(u,b))^{n-2}}{(n-2)!}e^{-\alpha_j C_n(u,b)}+$$

$$+\sum_{j=1}^{M}\frac{A_j^n}{2^n}\frac{(\alpha_j C_n(u,b))^{n-1}}{(n-1)!}e^{-\alpha_j C_n(u,b)}$$

Proof: It can be directly proved by the definition and Theorem 3.5.

Corollary 3.7: For any $b \geq u$, we define $T_b = \min(t \geq 0 : U(t) = b)$ to be the first time when the surplus reaches level b. Then the probability of the surplus attain a certain level b of $\Psi_n(u,b)$ satisfy the following formula when the claim distribution is double exponential.

$$\Psi_n(u,b) = \sum_{i=1}^{M} \frac{A_i}{2} e^{-\alpha_i C_1(u,b)} + \sum_{i=2}^{n} \sum_{k=0}^{i-2} \sum_{j=1}^{M} \frac{1}{2^i} A_j^{i-1} \frac{(\alpha_j C_i(u,b))^k}{k!} e^{-\alpha_j C_i(u,b)}$$
$$+ \sum_{i=2}^{n} \sum_{j=1}^{M} \frac{A_j^{i-1}(2 - A_j)}{2^{i+1}} \frac{(\alpha_j C_i(u,b))^{i-2}}{(i-2)!} e^{-\alpha_j C_i(u,b)} + \sum_{i=2}^{n} \sum_{j=1}^{M} \frac{A_j^i}{2^i} \frac{(\alpha_j C_i(u,b))^{i-1}}{(i-1)!} e^{-\alpha_j C_i(u,b)}$$

Proof: It can be directly proved by the Corollary 3.6.

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