# Procedure for Process Control with Inspection Errors

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## Abstract
Several papers have considered the procedure of online process control when the possibility of misclassification of items exists. In this paper, we consider online process control in which every $h^{\text{th}}$ item is inspected. The item is subjected to repeated independent classifications and will ultimately judged to be a conforming item if there are $k$ consecutive judgments that it is conforming prior to a $f$ consecutive judgments that it is nonconforming. When the item is ultimately judged to be nonconforming, the process is stopped and a search for a cause is conducted. If no cause is found, the process is put back online. When the item is judged as conforming, the process continues. There is a possibility of a misclassification of a conforming item as nonconforming and nonconforming item as conforming. We derived the probabilities of several quantities of interest for this model.

**Key Words:** Acceptance Sampling, Attribute, Misclassification, Process Control, Runs

### 1. Introduction

Taguchi, Elsayed, and Hsiang(1989) and Taguchi, Chowdhury, and Wu (2004) consider on-line process control by attributes involves inspecting every $h^{\text{th}}$ item produced. Initially the process is assumed to have some high fraction of conforming items, say 100 % or close to it. That is, an item conforms to specifications with probability $p_1$, equal to or very close to 1 when the process is in control. At some random time the process goes out of control and there is a shift to $p_2$ ($< p_1$) for the fraction conforming, or the probability that the selected item is really conforming. When an inspected item is considered nonconforming, the process is stopped for adjustment.

Variations on this theme have been introduced by a number of authors. Nayebpour and Woodall (1993) consider the random time until the shift from $p_1$ to $p_2$ to follow a geometric distribution. That is, the items produced are modeled as independent and identically distributed trials with a constant probability $\pi$ for each item to be the first item produced after the shift of the fraction conforming. Since only every $h^{th}$ item is inspected, the first item produced after the shift may not be inspected and thus there may be some initial number of items produced before the possibility of the detection of this shift even exists.

Borges, Ho, and Turnes (2001) argue that the inspection process itself may be subject to possible diagnostic errors. That is, in a single classification, a conforming item might be mistakenly classified as nonconforming, let $p_{CN}$ be the probability of this misclassification. In addition, a nonconforming item might mistakenly be judged as conforming, let $p_{NC}$ be the probability of this misclassification. We will also define probability $p_{CC}$ ($p_{NN}$) of the correct classification that a conforming (nonconforming) item is classified as conforming (nonconforming). This leads to the notion of making repeated classifications of each inspected item before making the final determination as to whether to judge the item as conforming or nonconforming. If the item has been judged in this final determination to be nonconforming, the process is judged out of control and is stopped for adjustment. Otherwise, the process is considered in control and is not stopped for adjustment. Because of the possibility of diagnostics errors in the repeated classifications, it is possible that an item is judged to be nonconforming and thus that the process is judged out of control, when it actually is not. Still it is stopped for adjustment. However, in that case, no cause can be found and the process then is restarted and has not, in some way, been put out of control by the stopping and searching for a cause. On the other hand, it is also possible that the process goes out of control, but is not detected. In that case, it remains out of control until this is detected at a later time, when it will be adjusted and be put back in control.

In Trindade, Ho, and Quinino (2007), the rule for the final determination of whether the inspected item is conforming, and thus whether the process is in control, is based on a pre-specified number of repeated classifications and using majority rule. Quinino, Colin, and Ho (2009) consider a rule in which the item is determined to be conforming and the process to be in control if and only if there are $k$ classifications as conforming before $f$ classifications as nonconforming, where $k$ and $f$ are some pre-specified positive integers. The acronym TCTN is used to describe this rule since the decision is based on the total number of classifications as conforming and nonconforming. Smith and Griffith (2009) studied this rule.

Smith and Griffith (2011) studied an alternative rule in which the final determination that an item is conforming, and thus the process is in control, if and only if $k$ consecutive classifications as conforming occur before a total of $f$ classifications as nonconforming. They used the acronym CCTN for this rule.

In this paper we will present and study an additional rule in which the final determination that an item is conforming, and thus the process is in control, if and only if $k$ consecutive classifications as conforming occur before a $f$ consecutive classifications as nonconforming. We will use the acronym CCCN for this rule.

## 2. Probabilistic Analysis

We propose some probabilistic questions relevant to the study of repeated classifications and provide the answers and the proofs in some instances.

1) Given that the item being inspected is conforming (nonconforming), what is the probability that it is judged to be conforming?
   ANSWER: Let $k$ be the number of consecutive conforming classifications and $f$ the consecutive number of nonconforming classifications needed for decision, then

$$P(judged\ conforming|\ conforming) = CCCN(p_{CC}) = \frac{p_{CC}^{k\text{-}1}\left[1\text{-}(1\text{-}p_{CC})^{f}\right]}{1\text{-}\left(1\text{-}p_{CC}^{k\text{-}1}\right)\left[1\text{-}(1\text{-}p_{CC})^{f\text{-}1}\right]}$$

$$P(judged\ conforming \mid nonconforming) = CCCN(p_{NC})$$
$$= \frac{p_{NC}^{k-1}\left[1-(1-p_{NC})^{f}\right]}{1-\left(1-p_{NC}^{k-1}\right)\left[1-(1-p_{NC})^{f-1}\right]}$$

PROOF:

Consider the Markov chain $\{X_n\}$ with state space

$$\{(r,s): 0 \le r \le k, s = 0\} \cup \{(r,s): r = 0, 0 \le s \le f\}$$

where $X_n = (r,s)$ means that after the $n^{th}$ classification there are $r$ consecutive successes and $s$ consecutive failures. Let $p_{CC}$ ( $p_{NC}$) be the probability that a conforming (nonconforming) item is classified as conforming. In the analysis below, we $p$ will be equal to $p_{CC}$ or $p_{NC}$ depending on the true nature of the item. The transition probabilities when beginning in a transient state are of the form

$$P(X_n = (r+1,0) \mid X_{n-1} = (r,s)) = p \text{ and } P(X_n = (0,s+1) \mid X_{n-1} = (r,s)) = q.$$

The situation is depicted in Figure 1.

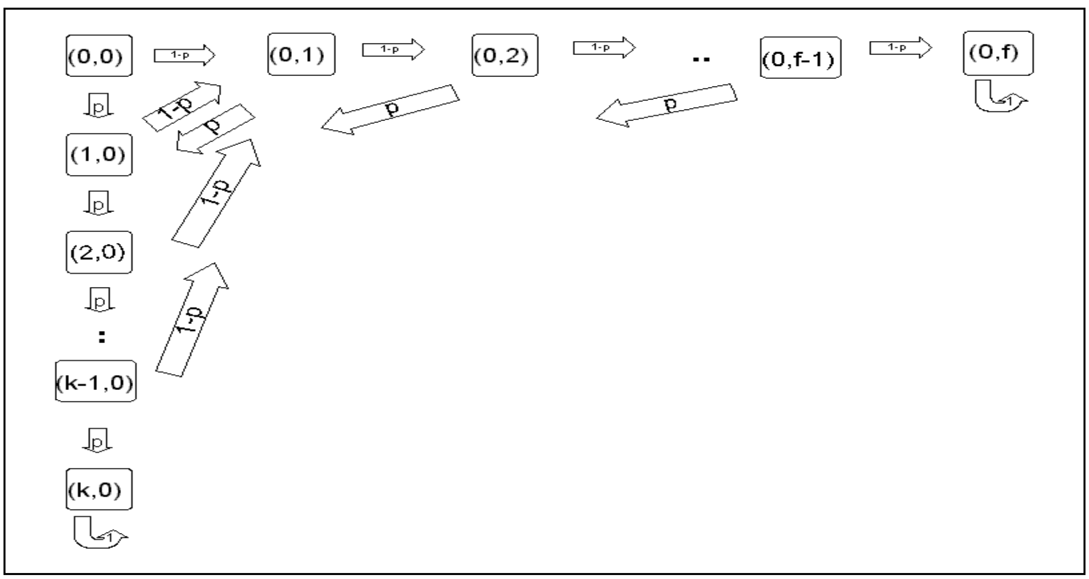


Figure 1.

Given that we are in the first column, we move down the column with probability $p$ and move to state (0,1) with probability 1-$p$. Given that we are in the first row we go across the row with probability 1-$p$ and move to state (1,0) with probability $p$. The state ($k$,0) is the absorbing state corresponding to item is judged conforming and state (0,$f$) is the absorbing state corresponding to item is judged nonconforming.

Consider figure 2 with a reduced state space and transition probabilities.

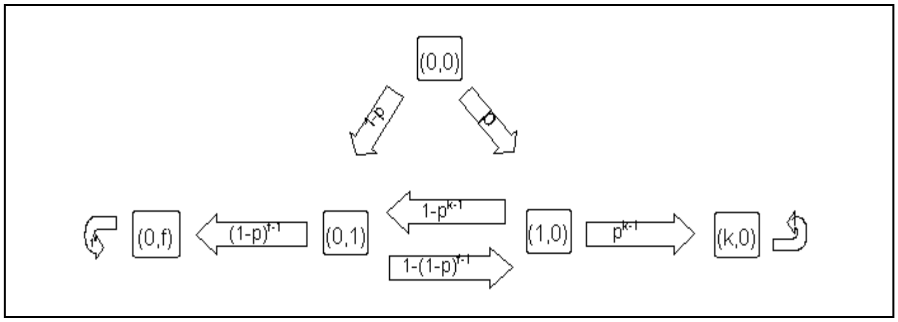


Figure 2.

For example, when the chain is in state (1,0), there are either $k$-1 consecutive successes (causing the chain to enter state ($k$, 0)) or there are not $k$-1 consecutive successes (causing the chain to enter state (0,1) upon a failure). Similarly, when the chain is in state (0,1), there are either $f$-1 consecutive failures (causing the chain to enter state (0,$f$)) or there are not $f$-1 consecutive failures (causing the chain to enter (1,0) upon a success).

To find the probability of judging the item to be conforming we may reason as follows. Starting in state (0,0), the chain enters state (1,0) with probability $p$ and state (0,1) with probability 1-$p$. If the process enters (1,0), it can eventually get to ($k$,0) by going directly, or by going to (0,1) and back to (1,0) any integer number of times and then to state ($k$,0) directly. Hence the probability of reaching state ($k$,0) from state (1,0) is

$$p^{k-1}+\sum_{n=1}^{\infty}\left[\left(1-p^{k-1}\right)\left(1-(1-p)^{f-1}\right)\right]^{n} p^{k-1}=\frac{p^{k-1}}{1-(1-p^{k-1})(1-(1-p)^{f-1})}$$

On the other hand, if the process enters (0,1), it can eventually get to state ($k$,0) only by going to state (1,0) (rather than state (0,$f$) which is absorbing). The probability of reaching state ($k$,0) from state (1,0) has been calculated above. Hence,

$$\begin{aligned}
&\mathrm{P(Judged\ Conforming)}\\
&\quad= p\cdot\frac{p^{k-1}}{1-(1-p^{k-1})(1-(1-p)^{f-1})}+(1-p)\left[1-(1-p)^{f-1}\right]\\
&\quad\cdot\frac{p^{k-1}}{1-(1-p^{k-1})(1-(1-p)^{f-1})}=\frac{p^{k-1}\left[1-(1-p)^{f}\right]}{1-(1-p^{k-1})(1-(1-p)^{f-1})}
\end{aligned}$$

2) Given that the process is in control, what is the probability that it is judged to be in control?

ANSWER: If it is in control, then the inspected item is conforming with probability $p_1$ and nonconforming with probability 1-$p_1$. In light of the answer to question 1 and using the law of total probability,

$$P_{II} = P(judged\ in\ control|\ in\ control) = p_1 CCCN(p_{CC}) + (1-p_1)CCCN(p_{NC})$$

3) Given that the process is out of control, what is the probability that is judged to be in control?
ANSWER: If out of control, then inspected item conforms with probability $p_2$ and fails to conform with probability $1 - p_2$. In light of the answer to question 1 and using the law of total probability,

$$\begin{aligned} P_{OI} &= P(judged\ in\ control|\ out\ of\ control) \\ &= p_2 CCCN(p_{CC}) + (1-p_2)CCCN(p_{NC}) \end{aligned}$$

4) Once it goes out of control, what is the distribution of the number of inspections needed to determine it is out of control?
ANSWER:
This is geometric distribution with parameter $\theta = 1 - P_{OI}$.

5) Let $Y$ = time measured in decision time until the process actually goes out of control and $\pi$ is the probability of a shift on any item produced then the
$P(Y = y) = [(1-\pi)^h]^{y-1}[1-(1-\pi)^h] = \theta(1-\theta)^{y-1}$, $y = 1, 2, 3, \ldots$ .
So Y has a geometric distribution with parameter $\theta = 1-(1-\pi)^h$ .

6) Let $X$ = time measured in decision time until the process is judged out of control.
$P(X = x) = \sum_{y=1}^{\infty} P(X = x|Y = y)P(Y = y)$ where

$$\mathrm{P}(X = x|Y = y) = \begin{cases} [P_{II}]^{x-1}[1-P_{II}], & x < y \\ [P_{II}]^{x-1}[1-P_{OI}], & x = y \\ [P_{II}]^{y-1}[P_{OI}]^{x-y}[1-P_{OI}], & x > y \end{cases}$$

$$\mathrm{P}(X = x) = \sum_{y=1}^{x-1}[P_{II}]^{y-1}[P_{OI}]^{x-y}\,[1-P_{OI}]\,(\theta(1-\theta)^{y-1}) +$$

$$[P_{II}]^{x-1}[1-P_{OI}](\theta(1-\theta)^{y-1}) + \sum_{y=x+1}^{\infty}[P_{II}]^{x-1}[1-P_{II}](\theta(1-\theta)^{y-1})$$

7) Given that the item being inspected is conforming, what is the mean, variance, and probability mass function of the time until a decision is reached? What if the item is nonconforming?

ANSWER: Consider the Markov Chain $\{X_n\}$ with state space

$$\{(r,s): 0 \le r \le k, s = 0\} \cup \{(r,s): r = 0, 0 \le s \le f\}$$

where $X_n = (r,s)$ means that after the $n^{\text{th}}$ classification there are $r$ consecutive conforming and $s$ consecutive nonconforming. In the following we will use $p = p_{\text{CC}}$ if we are given the item is conforming and use $p = p_{\text{NC}}$ if we are given the item is nonconforming. The transition probabilities are of the form

$$\mathrm{P}(X_n = (r+1,0) \mid X_{n-1} = (r,s)) = p \text{ and } \mathrm{P}(X_n = (0,s+1) \mid X_{n-1} = (r,s)) = q.$$

For each Markov chain there are absorbing (recurrent) states, which correspond to the end of the repetitive classifications for a single item and the consequent decision. Let $A$ denote the set of absorbing states and $a$ denote the number of absorbing states. In fact, the singleton sets consisting of each of these absorbing states are recurrent classes. The remaining states are transient which we will denote by T and likewise the number of transient states by $t$. Written in canonical form, the one-step transition probability matrix $\mathbf{P}$ for the Markov chain is $\begin{bmatrix} \mathbf{P_1} & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$, where $\mathbf{P_1}$ is the $a \times a$ identity matrix for the absorbing states, $\mathbf{R}$ is a $t \times a$ matrix containing the one-step probabilities of the transient states to the recurrent (absorbing) states, $\mathbf{Q}$ is a $t \times t$ matrix containing the one-step probabilities among the transient states, and $\mathbf{0}$ is the $a \times t$ zero matrix. The one-step probabilities of $\mathbf{R}$ and $\mathbf{Q}$ are determined by the transition probabilities given for each test. The first row of $\mathbf{Q}$ contains the one step transition probabilities from state (0,0).

To compute the moments of the decision time, we will define the following notation. Since elements of T appear as subscripts, we will use $i$ and $j$ as typical elements of T. However, it should be noted that when we do so, each of $i$ and $j$ refer to an ordered pair such as $(r,s)$. Let,

- $\mathbf{I}_{t \times t}$ = identity matrix of dimension $t \times t$
- $\mathbf{M}_{t \times t} = (\mathbf{I}_{t \times t} - \mathbf{Q}_{t \times t})^{-1}$ - the fundamental matrix of dimension $t \times t$
- $\mathbf{e}_m$ = column vector of length $t$ where the $m^{\text{th}}$ element is one and the remaining elements are zero.
- $\boldsymbol{e_m}'$ is defined to be the transpose of $\mathbf{e}_m$
- $\mathbf{u}_{\{RS\}}$ = column vector where all the elements corresponding to the rejection states are one, and the remainder of the elements are zero.
- $\mathbf{1}_z$ = column vector of ones of length $z$
- $N_{ij}$ = random variable that represents the number of times the process visits state $j$ before it eventually enters a recurrent state, having initially started from state $i$ $(i,j \in \mathrm{T})$.
- $\mu_{ij} = \mathrm{E}(N_{ij})$ for $i,j \in \mathrm{T}$.
- $\mathbf{M}_\rho = \left[\sum_{j \in T} \mu_{ij}\right] = \mathbf{M}\mathbf{1}_t$ - column vector such that the $m^{\text{th}}$ element is the sum of the $m^{\text{th}}$ row of $\mathbf{M}$
- $\mathbf{M}_{\rho^2} = \left[\left(\sum_{j \in T} \mu_{ij}\right)^2\right] = diag(\mathbf{M}_{\boldsymbol{\rho}})\mathbf{M}_{\boldsymbol{\rho}}$ - column vector such that the $m^{\text{th}}$ element is the square of the sum of the $m^{\text{th}}$ row of $\mathbf{M}$. Note: diag $(\mathbf{M}_\rho)$ is a diagonal matrix whose entries are the corresponding entries of $\mathbf{M}_\rho$.

The results below are given without proof and based on formulas in Bhat (1984).

Consider the decision time for a single item for i.i.d. Bernoulli classifications with constant probability of conforming, let $p = p_{CC}$ and with the first transient state being the initial state.

a) Expected decision time
E(Decision time |conforming) = $\mathbf{e}_1'\mathbf{M1}_t$. where $p = p_{CC}$

E(Decision time |nonconforming) = $\mathbf{e}_1'\mathbf{M1}_t$. where $p = p_{NC}$

E(Decision time|in control) = $p_1$ E(Decision time |conforming) + (1-$p_1$) E(Decision time |nonconforming)

E(Decision time|out of control) = $p_2$ E(Decision time |conforming) + (1-$p_2$) E(Decision time |nonconforming)

b) The variance of decision time
Var(Decision time |conforming) =$e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}} - \mathbf{M}_{\rho^2}\right]$.
where $p = p_{CC}$
Var(Decision time |nonconforming) = $e_1'\left[(2\mathbf{M}-\mathbf{I})\mathbf{M}_{\boldsymbol{\rho}} - \mathbf{M}_{\rho^2}\right]$.
where $p = p_{NC}$

c) The probability mass function of the decision time
P(decision time $= m$ | conforming) = $\mathbf{e}_1' \mathbf{Q}^{m-1}\mathbf{R}\, \mathbf{1}_a$. where $p = p_{CC}$

P(decision time $= m$ | nonconforming) = $\mathbf{e}_1' \mathbf{Q}^{m-1}\mathbf{R}\, \mathbf{1}_a$. where $p = p_{NC}$

8) We can determine the minimum values of $k$ & $f$ that should be used to insure that the that P(judged in control | process is out of control) < α and the P(judged in control | process is in control) > 1- β.

EXAMPLE: Let $p_1 = 0.95$, $p_2 = 0.10$, $p_{CC} = 0.8$, $p_{NC} = 0.2$, $\alpha = 0.10$, $\beta = 0.20$
ANSWER: $k = 5$ and $f = 3$.
In addition, for this example:
E(Decision time |conforming) = 9.6220
E(Decision time |nonconforming) = 4.7598
E(Decision time| in control) = 0.95 (9.6220) + (1-0.95) 4.7598 = 9.3788
E(Decision time| out of control) = 0.10 (9.6220) + (1-0.10) 4.7598 = 5.2460
Var(Decision time |conforming) = 36.8556
Var(Decision time |nonconforming) = 6.9416
CCTN ($p_{CC}$) = P(judged conforming | conforming) = 0.9379
CCTN ($p_{NC}$) = P(Judged conforming | nonconforming) = 0.0012
$P_{II}$ = P(judged in control | in control) = $p_1$ CCTN ($p_{CC}$) + (1 – $p_1$) CCTN ($p_{NC}$)
= 0.95 (0.9379) + (1-0.95) 0.0012 = 0.8911

$P_{OI}$ = P(judged in control | out of control) = $p_2$ CCTN ($p_{CC}$) + (1 – $p_2$) CCTN ($p_{NC}$)
= 0.10 (0.9379) + (1-0.10) 0.0012 = 0.0957

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