A Game Theoretic Approach To Risk Management

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Abstract
Risk management has traditionally been quantified by an expert-led assessment of the states of a process, the likelihood and consequences of being in each state at a particular time or through a given sequence of previous states. Such an approach is oriented to an ever-changing set of conditions and is evaluated by the practical experience of the assessment team. As senior management sets the policies to address the issues raised by the risk assessment, its continual reconsideration of those policies as circumstances change puts an uncertain burden on those responsible for risk abatement. Unless continual risk reassessment and abatement revision is made part of its quality culture, a management team does not make optimal use of its risk management potential.

However, rather than simply follow the lead promoted by classical risk assessment processes, an alternative approach might be developed. One of the alternatives is a game theoretic approach based on a model of "public" and "menace" activities. Such an approach could mathematically and dynamically chose a strategy for minimizing risk based on current resources and the state of the process. Such models consider all changes in process states simultaneously, so that an optimal strategy may be chosen before actual changes are observed. Game theoretic models of risk management are adaptive, informative, and guiding in nature. They eliminate the need for a preponderance of specific historical data, as the methods for choosing a strategy are not particularly based on what has happened to date, rather on what might happen in the future.

The purpose of this paper is to present the nomenclature and elementary theory for such a game theoretic application to risk management. It may be applied to any process where the competing interests of a structured two-person zero-sum interactive game with random chance elements may be applicable.

Key Words: Game Theory, Risk Management, Reduction Equivalence

1. Introduction

A **game** may be thought of as a finite set of choices made by several **players** or random moves of chance made in place of a player's choice, each choice being made among the players or through a chance process simultaneously, e.g., all first round choices are made before any second round choices are made, and independently, i.e., no player is allowed to change a choice once made. Sometimes a player will make a choice in the presence of some **information** on the prior state of other players' choices. That information is strictly based on what *has* happened and what *could* happen next, and is not based on what another player *will* do or *has* made for the next choice. A collection of choices a player makes in the game is called a **strategy** and the set of all possible strategies available to a player in a game is called a **strategy set**. Each strategy pursued by a player during a game results in an **outcome** which is evaluated by a **utility function** to assess to what extent the outcome is desirable. For each player, the **objective** of the game is to chose the strategy that results in the most desirable outcome as measured by the utility function.

To make these definitions precise, and to establish the theoretical grounds on which an application to risk management will be developed, the following nomenclature will be used.

1. A **tactical choice** is a (non-random) instruction for a move on the n^{th} round in a game after all other $(n-1)$ round moves by all players have been made. The n^{th} round move of player i is written $f_i(n)$. The totality of all such tactical choices for a player for the whole game is called a **strategy**.
2. A **strategy set** F is the space of all possible strategies f for a given player.
3. A **chance move** is an instruction in a game that is made by a random process rather than by a player's choice.

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4. A **circumstance** w is the history of the chance and players' previous moves that have resulted in a particular situation in the game. It is at this point that a chance move or a tactical choice needs to be made by a random process or by the players, respectively.
5. The **circumstance set** W is the space of all possible circumstances when a chance move is required.
6. The **choice set** $S(w)$ is the set of tactical choices a player may choose at circumstance w in the game.
7. The set of **information sets** $\{V(w)\}$ of a circumstance w is a partition of $S(w)$ such that a player with choices in $V(w)$ only knows $V(w)$ and not w.
8. The **chance function** $h : W \to \bigcup_w S(w)$ specifies for each circumstance involving a chance move which choice is to be selected. Here, $h(w) \in S(w)$.
9. The **chance space** K is the space of all chance functions defined on W into $\bigcup_w S(w)$.
10. A k**-player game** g (or simply a **game** if the number of players is not at issue) is the $(k+1)$-tuple $g = (f_1, f_2, ..., f_k, h)$, where $f_i \in F_i$, and $h \in \mathcal{K}$.
11. The **power game** $\mathcal{G} = (F_1, F_2, ..., F_k, h, u)$ is the set of all possible games for a given collection of choice sets S, circumstance set W, and utility function u.
12. The **chance distribution** P is the probability distribution function by which h makes its choices. Since P does not depend on, nor is influenced by, the $(f_1, f_2, ..., f_k)$ in g, P may be thought of as a probability distribution on $\mathcal{G}$. The chance distribution is written P_f when viewed in this manner.
13. The **outcome function** $q : \mathcal{G} \to \mathcal{R}$ returns a unique **outcome** r for a given k-player game g, i.e., $q(g) = r$.
14. When two or more players' tactical instructions in a circumstance are coordinated to achieve a particular outcome, then the players are said to have engaged in **collusion**.
15. The **outcome distribution** π_f is the extension of P_f through q to $\mathcal{R}$.
16. The **utility function** $u : \mathcal{R} \to \mathbb{R}$ measures the extent to which an outcome is desirable, i.e., outcome r_1 is considered more desirable than outcome r_2 if $u(r_1) > u(r_2)$, and $u(r_1) = u(r_2)$ means an evaluator is indifferent to outcomes r_1 and r_2. The value of $u(r)$ is called the **utility** of outcome r.
17. The **objective** of a game for a player is to pursue a strategy whose outcome has a desired utility.
18. A **game in normal form** is a k-player game without collusion with strategy sets $F_1, F_2, ..., F_k$ (F_i corresponds to player i strategies), chance function h, outcome distribution π_f, and k-many numerical functions $M_i : \left(\prod_{j=1}^{k} F_j, h\right) \to \mathbb{R}$, such that $M_i(f_1, f_2, ..., f_k, h)$ is the **payout** to player i. The outcomes of this kind of game are the payouts, and the utility function is the relative size of the payout, i.e., larger positive payouts are more desirable than smaller positive payouts, and smaller negative payouts are more desirable than larger negative payouts. The objective of the game for each player is to maximize the payout.
19. A **two-person zero-sum (power) game** $G = (f_1, f_2, h, M)$ is a game in normal form where $k = 2$ and $\sum_{i=1}^{2} M_i(f_1, f_2, h) = 0$, i.e., "one player's loss is the other player's gain." The function M is the absolute value of either M_1 or M_2, since they only differ by sign in this case. The term "zero-sum" is not necessarily associated with any particular number of players, as long as the sum of all payouts is zero. Any such game is called a **power game** when the structure of G is being emphasized.

2. Game Examples

Several examples of simple games will help illustrate the terms defined in the previous section.

2.1 Casino Games

All casino gambling may be viewed as two-person zero-sum games. The two players are "you" and "the house." The tactical choices depend on the game, such as whether to "hit" a hand in blackjack, or "place odds" in craps. The information sets differ by which cards have been exposed in blackjack; however, there is only one information set in craps. The outcomes are "win," "lose," or "nothing happens," and the payouts depend on the amount bet and the rules of the game. The chance distribution is set by the use of random processes (dice, spinning wheel, etc.), as is the outcome distribution. Your loss is the house's gain, and your gain is the house's loss. The objective of both you and the house is to win as much as possible, and you both measure this at any time by the amount of money you possess in excess of the amount you had at the start of the game.

2.2 Chance Element Game

A more explicit and complicated example is also a two-person zero-sum game, but with more intriguing interactions. The progress of this game is as follows:

1. Player I chooses first and selects one of the two integers 1 or 2.
2. A third party introduces an unbiased random process to choose either "left" or "right." If this chance move is "left," the third party informs Player II of Player I's choice. If the chance move is "right," the third party says nothing to Player II.
3. Player II then selects one of the two integers 3 or 4.
4. The third party introduces a chance move by selecting one of three integers 1, or 2, or 3, with respective probabilities 40%, 20%, and 40%.
5. The numbers in Steps 1, 3, and 4 are added together.
6. If the number in Step 5 is even, Player II pays Player I that amount in dollars. If the number in Step 5 is odd, Player I pays Player II that amount in dollars.

Certain parameter values are clear from this description: $k = 2$, $F_1 = \{\{1\}, \{2\}\}$, and

$$F_2 = \{\{i, j, k\}\}$$

where each i, j, k may be 3 or 4. The first position i refers to the information set described as "random process chooses 'left' and Player I chooses 1." The second position j refers to the information set described as "random process chooses 'left' and Player I chooses 2." The third position k refers to the information set described as "random process chooses 'right'." These sets refer strictly to Player II. A complete listing of F_2 would then be

$$F_2 = \{\{3,3,3\}, \{3,3,4\}, \{3,4,3\}, \{3,4,4\}, \{4,3,3\}, \{4,3,4\}, \{4,4,3\}, \{4,4,4\}\}$$

Hence, the choice sets $S = \{\{f_1 \in F_1, f_2 \in F_2\}\}$ across all such combinations of f_1 and f_2. The first chance function is $h_1 \in \{\text{"left"},\text{"right"}\}$ with chance distribution $P_1(\text{"left"}) = \frac{1}{2} = P_2(\text{"right"})$ (because the choice was unbiased), and the second chance function is $h_2 \in \{1, 2, 3\}$ with chance distribution $P_2(1) = 0.4$, $P_2(2) = 0.2$, and $P_2(3) = 0.4$. This means the joint chance function is $h = \{h_1, h_2\}$, and the chance distribution P_f may be expressed as

$$P(\{\text{"left"}, 1\}) = P(\{\text{"right"}, 1\}) = \left(\frac{1}{2}\right)(0.4) = 0.2$$
$$P(\{\text{"left"}, 2\}) = P(\{\text{"right"}, 2\}) = \left(\frac{1}{2}\right)(0.2) = 0.1$$
$$P(\{\text{"left"}, 3\}) = P(\{\text{"right"}, 3\}) = \left(\frac{1}{2}\right)(0.4) = 0.2$$

Hence, the circumstance set $W = \{\{h_1(w) \in \{\text{"left"},\text{"right"}\}, h_2(w) \in \{1, 2, 3\}\}\}$ across all circumstances w that involve a chance move.

The outcomes $r \in \mathcal{R}$ are either \$5, when, for example, Player I chooses 1, Player II chooses 3, and the second chance move chooses 1; or \$6 when, for example, Player I chooses 2, Player II chooses 3, and the second chance move chooses 1, continuing to $9 when, for example, Player I chooses 2, Player II chooses 4, and the second chance move chooses 3. There are $2 * 8 * 6 = 96$ games $g = (f_1, f_2, h)$, and therefore the outcome function has 96 parts:

$$q\left((f_1, f_2, h)\right) \in \{\$5, \$6, \$7, \$8, \$9\}$$

For example, $q\left(\left(\{1\}, \{3,4,3\}, \{\text{“left”}, 3\}\right)\right) = \$ \underbrace{(1+3+3)}_{\substack{h \text{ chooses } \{\text{“left”}, 3\} \in W \\ \text{I chooses 1} \\ \text{II chooses 3}}} = \$7.$

Since this amount is odd, from a utility point of view, the payout is

$$M_1\left(\left(\{1\}, \{3,4,3\}, \{\text{“left”}, 3\}\right)\right) = -\$7$$

from Player I's perspective, and it would be

$$M_2\left(\left(\{1\}, \{3,4,3\}, \{\text{“left”}, 3\}\right)\right) = +\$7$$

from Player II's perspective.

The complete computation of the outcome distribution $\pi_f(r)$ now follows. The h values are first classified by outcome source (see Figure 1).

	r				
$\{f_1, f_2\}$	\$5	\$6	\$7	\$8	\$9
$(\{1\}, \{3,3,3\})$	$L1R1$	$L2R2$	$L3R3$		
$(\{1\}, \{3,3,4\})$	$L1$	$L2R1$	$L3R2$	$R3$	
$(\{1\}, \{3,4,3\})$	$L1R1$	$L2R2$	$L3R3$		
$(\{1\}, \{3,4,4\})$	$L1$	$L2R1$	$L3R2$	$R3$	
$(\{1\}, \{4,3,3\})$	$R1$	$L1R2$	$L2R3$	$L3$	
$(\{1\}, \{4,3,4\})$		$L1R1$	$L2R2$	$L3R3$	
$(\{1\}, \{4,4,3\})$	$R1$	$L1R2$	$L2R3$	$L3$	
$(\{1\}, \{4,4,4\})$		$L1R1$	$L2R2$	$L3R3$	
$(\{2\}, \{3,3,3\})$		$L1R1$	$L2R2$	$L3R3$	
$(\{2\}, \{3,3,4\})$		$L1$	$L2R1$	$L3R2$	$R3$
$(\{2\}, \{3,4,3\})$		$R1$	$L1R2$	$L2R3$	$L3$
$(\{2\}, \{3,4,4\})$			$L1R1$	$L2R2$	$L3R3$
$(\{2\}, \{4,3,3\})$		$L1R1$	$L2R2$	$L3R3$	
$(\{2\}, \{4,3,4\})$		$L1$	$L2R1$	$L3R2$	$R3$
$(\{2\}, \{4,4,3\})$		$R1$	$L1R2$	$L2R3$	$L3$
$(\{2\}, \{4,4,4\})$			$L1R1$	$L2R2$	$L3R3$

Figure 1: Outcome Distribution $\pi_f(r)$

Here, R corresponds to “right,” and L to “left.” Entries with multiple letter/number combinations mean that the two individual values apply to the situation involved. Blank entries corresponds to a combination of $\{f_1, f_2\}$ and r that does not occur. The respective choices for h_2 are evident, and, for the remainder of the calculation detailed below, it will not be explicitly listed.

Since $P(L1) = P(R1) = 0.2$, $P(L2) = P(R2) = 0.1$, and $P(L3) = P(R3) = 0.2$, and all choices have been made independently at each step, then $\pi_f(r)$ may be expressed as in Figure 2.

This simplifies to the form found in Figure 3. We then may calculate the expected gain for Player I, keeping in mind the utility function signs (see Figure 4).

Finally, we have the final form of the payout function in Figure 5 (from the perspective of Player I).

The minimax strategy for Player I would be $\{1\}$, since this minimizes the maximum loss (the maximum loss under $\{1\}$ is $-\$3.60$, while under $\{2\}$ it is $-\$4.80$). The “best” strategy for Player I, i.e., the strategy that makes the maximum gain possible with the minimum loss potential, would be $\{1\}$, since the maximum gain (\$4.20) is the same as for $\{2\}$, yet the maximum loss is not as high for $\{1\}$, namely $-\$3.60$, than for $\{2\}$, namely $-\$4.80$. The average expected gain for strategy $\{1\}$ for Player I, assuming a uniform distribution on the choice of strategy for Player II, is \$0.30, while it is $-\$0.30$ for $\{2\}$.

	r				
$\{f_1, f_2\}$	\$5	\$6	\$7	\$8	\$9
$(\{1\}, \{3,3,3\})$	$0.2+0.2$	$0.1+0.1$	$0.2+0.2$		
$(\{1\}, \{3,3,4\})$	0.2	$0.1+0.2$	$0.2+0.1$	0.2	
$(\{1\}, \{3,4,3\})$	$0.2+0.2$	$0.1+0.1$	$0.2+0.2$		
$(\{1\}, \{3,4,4\})$	0.2	$0.1+0.2$	$0.2+0.1$	0.2	
$(\{1\}, \{4,3,3\})$	0.2	$0.2+0.1$	$0.1+0.2$	0.2	
$(\{1\}, \{4,3,4\})$		$0.2+0.2$	$0.1+0.1$	$0.2+0.2$	
$(\{1\}, \{4,4,3\})$	0.2	$0.2+0.1$	$0.1+0.2$	0.2	
$(\{1\}, \{4,4,4\})$		$0.2+0.2$	$0.1+0.1$	$0.2+0.2$	
$(\{2\}, \{3,3,3\})$		$0.2+0.2$	$0.1+0.1$	$0.2+0.2$	
$(\{2\}, \{3,3,4\})$		0.2	$0.1+0.2$	$0.2+0.1$	0.2
$(\{2\}, \{3,4,3\})$		0.2	$0.2+0.1$	$0.1+0.2$	0.2
$(\{2\}, \{3,4,4\})$			$0.2+0.2$	$0.1+0.1$	$0.2+0.2$
$(\{2\}, \{4,3,3\})$		$0.2+0.2$	$0.1+0.1$	$0.2+0.2$	
$(\{2\}, \{4,3,4\})$		0.2	$0.1+0.2$	$0.2+0.1$	0.2
$(\{2\}, \{4,4,3\})$		0.2	$0.2+0.1$	$0.1+0.2$	0.2
$(\{2\}, \{4,4,4\})$			$0.2+0.2$	$0.1+0.1$	$0.2+0.2$

Figure 2: Re-expressed $\pi_f(r)$

	r				
$\{f_1, f_2\}$	\$5	\$6	\$7	\$8	\$9
$(\{1\}, \{3,3,3\})$	0.4	0.2	0.4		
$(\{1\}, \{3,3,4\})$	0.2	0.3	0.3	0.2	
$(\{1\}, \{3,4,3\})$	0.4	0.2	0.4		
$(\{1\}, \{3,4,4\})$	0.2	0.3	0.3	0.2	
$(\{1\}, \{4,3,3\})$	0.2	0.3	0.3	0.2	
$(\{1\}, \{4,3,4\})$		0.4	0.2	0.4	
$(\{1\}, \{4,4,3\})$	0.2	0.3	0.3	0.2	
$(\{1\}, \{4,4,4\})$		0.4	0.2	0.4	
$(\{2\}, \{3,3,3\})$		0.4	0.2	0.4	
$(\{2\}, \{3,3,4\})$		0.2	0.3	0.3	0.2
$(\{2\}, \{3,4,3\})$		0.2	0.3	0.3	0.2
$(\{2\}, \{3,4,4\})$			0.4	0.2	0.4
$(\{2\}, \{4,3,3\})$		0.4	0.2	0.4	
$(\{2\}, \{4,3,4\})$		0.2	0.3	0.3	0.2
$(\{2\}, \{4,4,3\})$		0.2	0.3	0.3	0.2
$(\{2\}, \{4,4,4\})$			0.4	0.2	0.4

Figure 3: Simplified $\pi_f(r)$

These three analyses argue that $\{1\}$ is the optimal strategy for Player I. The "best" strategy for Player II is $\{3, 4, 3\}$, since it guarantees an expected gain, i.e., either \$3.60 if Player I chooses $\{1\}$, and \$0.30 if Player I chooses $\{2\}$. Recall that the payout for Player II is the negative of the payout for Player I. The best strategy for Player II for the benefit of Player I is $\{4, 3, 4\}$, since Player I would expect to only lose \$0.30 (the minimum loss for both Player I strategies) if Player 1 chooses $\{2\}$, and would expect to gain \$4.20 (the maximum gain for both Player I strategies) if Player I chooses $\{1\}$.

2.3 Stick or Change Game

Another example of a two-person zero-sum game with chance elements is modeled on the "Let's Make A Deal" game show. The progress of the game is as follows:

1. A random process chooses one of the numbers $1, 2, 3$.
2. Player I chooses one of the numbers $1, 2, 3$.
3. Player II is informed of both Player I's choice and the choice of the random process. If Player I has chosen the same number as was chosen by the random process, then Player II unbiasedly chooses one of the other two numbers and announces to Player I that Player II's number is not the same as the choice from the random process. If Player I has chosen a number different than the random process, then Player II chooses the remaining number not chosen by Player I or the random process, and announces to Player I that Player II's number is not the same as the choice from the random process.

	f_2			
f_1	$\{3,3,3\}$	$\{3,3,4\}$	$\{3,4,3\}$	$\{3,4,4\}$
$\{1\}$	$6*.2$ $-5*.4$ $-7*.4$	$6*.3$ $-5*.2$ $-7*.3$ $+8*.2$	$6*.2$ $-5*.4$ $-7*.4$	$6*.3$ $-5*.2$ $-7*.3$ $+8*.2$
$\{2\}$	$6*.4$ $-7*.2$ $+8*.4$	$6*.2$ $-7*.3$ $+8*.3$ $-9*.2$	$6*.2$ $-7*.3$ $+8*.3$ $-9*.2$	$8*.2$ $-7*.4$ $-9*.4$
	f_2			
f_1	$\{4,3,3\}$	$\{4,3,4\}$	$\{4,4,3\}$	$\{4,4,4\}$
$\{1\}$	$6*.3$ $-5*.2$ $-7*.3$ $+8*.2$	$6*.4$ $-7*.2$ $+8*.4$	$6*.3$ $-5*.2$ $-7*.3$ $+8*.2$	$6*.4$ $-7*.2$ $+8*.4$
$\{2\}$	$6*.4$ $-7*.2$ $+8*.4$	$6*.2$ $-7*.3$ $+8*.3$ $-9*.2$	$6*.2$ $-7*.3$ $+8*.3$ $-9*.2$	$8*.2$ $-7*.4$ $-9*.4$

Figure 4: Expected Gain For Player I

	f_2							
f_1	$\{3,3,3\}$	$\{3,3,4\}$	$\{3,4,3\}$	$\{3,4,4\}$	$\{4,3,3\}$	$\{4,3,4\}$	$\{4,4,3\}$	$\{4,4,4\}$
$\{1\}$	**−\$3.60**	**\$0.30**	**−\$3.60**	**\$0.30**	**\$0.30**	**\$4.20**	**\$0.30**	**\$4.20**
$\{2\}$	**\$4.20**	**−\$0.30**	**−\$0.30**	**−\$4.80**	**\$4.20**	**−\$0.30**	**−\$0.30**	**−\$4.80**

Figure 5: Final Form Of Expected Gain For Player I

4. Player I then chooses to either "stick" with his original choice, or "change" to the remaining number not chosen by Player II.
5. If Player I's final choice is the same as that chosen by the random process, Player II pays Player I \$1. Otherwise, Player I pays Player II \$1.

This is a two-person zero-sum game without collusion. This game introduces a complication not found in the previous examples. In those games, the choice sets $S(w)$ did not depend on the circumstances w for each player. In the present game, the choice set for Player II at Step 3 depends on the chance move in Step 1 *and* the choice of Player I in Step 2. To make the nomenclature clear, generic choices are labeled with letters, and explicit choices are labeled by numerals.

The chance function in Step 1 would be $h \in \{\{1\},\{2\},\{3\}\}$, and let us assume a uniform distribution of such choices:

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = \frac{1}{3}.$$

There is only one information set for Player I at Step 2, so $F_1^{\text{Step 2}} = \{\{1\},\{2\},\{3\}\}$. The information sets for Player II at Step 3 are (1) "Player I has chosen the same number as the random process," and (2) "Player I has not chosen the same number as the random process." Therefore, according to the rules of the game, $F_2 = \{\{i,j\}\}$, where i is the choice under the first information set, and j is the choice under the second information set. The strategic choices for i are $\{C1, C2\}$, where $C1$ means "choose one of the other two numbers" and $C2$ means "choose the other one of the other two numbers." The strategic choices for j are $\{\{k\}\}$, where k is the remaining number not yet chosen. This means

$$F_2 = \{\{C1,k\},\{C2,k\}\}.$$

There is only one information set for Player I at Step 4: Player II has announced that one of the other two numbers not chosen by Player I in Step 2 is not the same as that chosen by the random process. The strategic choices for Player I at this point are $F_1^{\text{Step 4}} = \{\{S\},\{C\}\}$, where S represents "stick with the original choice," C means "change to the third number not chosen previously." So a complete enumeration of F_1 and F_2 would be

$$F_1 = \{\{1,S\},\{1,C\},\{2,S\},\{2,C\},\{3,S\},\{3,C\}\}$$

and

$$F_2 = \{\{C1, k\}, \{C2, k\}\}$$

where k depends on the choice made by Player I in Step 2.

The expected payout matrix from the perspective of Player I would then be calculated as in Figure 6 if h chooses $\{1\}$, with probability $\frac{1}{3}$.

	f_2								
f_1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$
$\{1,S\}$				$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\{1,C\}$				$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$\{2,S\}$			$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$
$\{2,C\}$			$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$
$\{3,S\}$		$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$	
$\{3,C\}$		$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$	

Figure 6: Expected Payout Matrix For Player I If h Chooses$\{1\}$

The term $\frac{1}{6}$ comes from one-third of $\frac{1}{2}$, since the choice between $C1$ and $C2$ is unbiased. The expected payout matrix would be calculated as in Figure 7 if h chooses $\{2\}$, with probability $\frac{1}{3}$. Finally, the expected payout matrix would be calculated as in Figure 8 if h

	f_2								
f_1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$
$\{1,S\}$			$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$
$\{1,C\}$			$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$
$\{2,S\}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$				$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\{2,C\}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$				$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$\{3,S\}$	$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$		
$\{3,C\}$	$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$		

Figure 7: Expected Payout Matrix For Player I If h Chooses$\{2\}$

chooses $\{3\}$, with probability $\frac{1}{3}$. Combining these results gives the final expected payout

	f_2								
f_1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$
$\{1,S\}$		$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$	
$\{1,C\}$		$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$	
$\{2,S\}$	$-\frac{1}{3}$			$-\frac{1}{3}$			$-\frac{1}{3}$		
$\{2,C\}$	$\frac{1}{3}$			$\frac{1}{3}$			$\frac{1}{3}$		
$\{3,S\}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$			
$\{3,C\}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$			

Figure 8: Expected Payout Matrix For Player I If h Chooses$\{3\}$

matrix in Figure 9.

Each row of this matrix adds up to either $+1$ or -1; all $+1$ values correspond to $\{*, C\}$ strategies for Player I, and all -1 values correspond to $\{*, S\}$ strategies. Even though Player I could lose the game under certain circumstances pursuing a $\{*, C\}$ strategy, e.g., $(f_1, f_2) = \{\{2, C\}, \{3, 1\}\}$ would lose for Player I if $h = \{2\}$, Player I would expect to receive a higher payout than Player II under such strategies. Therefore, the expected winning policy for Player I is to change choices every time in Step 4 to whichever number has yet to be chosen. The expected payout of this policy is $\$\frac{1}{3}$, regardless of the strategy chosen by Player II. For example, if $f_2 = \{1, 3\}$, the expected payout under policy $\{*, C\}$ for Player I is $\frac{1}{3} + \frac{1}{6} - \frac{1}{6} = \frac{1}{3}$ (in dollars).

Intuitively speaking, the expected winning policy for Player I makes sense. The initial choice of Player I is "wrong," i.e., does not match the random process choice, with probability $\frac{2}{3}$. Player II effectively eliminates another "wrong" answer no matter the "correctness" of Player I's initial choice. Since Player I's initial choice was most likely wrong anyway, Player II has conveniently pointed the way to the matching answer. Even though Player I might have stumbled onto the matching answer from the beginning, and therefore

	f_2								
f_1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$
$\{1,S\}$		$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$
$\{1,C\}$		$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\{2,S\}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{3}$		$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$
$\{2,C\}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$		$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
$\{3,S\}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$	
$\{3,C\}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	

Figure 9: Final Form Of Expected Payout Matrix For Player I

changed to a "wrong" choice in Step 4, this is half as likely as otherwise. The calculations above justify these interpretations.

Each column of this matrix adds up to zero. This means there is no expected winning or losing strategy for Player II.

Note how this game would be significantly different if Player II were allowed a "no comment" strategic choice in Step 3. Then Player I would not have a guaranteed win if the first choice was "wrong," since Player II could simply offer no additional information to Player I in the next step. Then Player I would not only have to choose to stick or to change, but to which number the change should be.

If all choices by Player II are unbiased, the expected payout matrix would be calculated as in Figure 10, where N means "no comment." Now the situation is very different. Every

	f_2							
f_1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,N\}$	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{2,N\}$
$\{1,1\}$		$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{6}$
$\{1,2\}$			$\frac{1}{8}$	$-\frac{1}{8}$			$\frac{1}{8}$	$-\frac{1}{8}$
$\{1,3\}$		$\frac{1}{8}$		$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{2,1\}$			$\frac{1}{8}$	$-\frac{1}{8}$			$\frac{1}{8}$	$-\frac{1}{8}$
$\{2,2\}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{6}$	$-\frac{1}{8}$		$-\frac{1}{8}$	$-\frac{1}{4}$
$\{2,3\}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$			$-\frac{1}{8}$
$\{3,1\}$		$\frac{1}{8}$		$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{3,2\}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{8}$			$-\frac{1}{8}$
$\{3,3\}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{6}$

	f_2							
f_1	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$	$\{3,N\}$	$\{N,1\}$	$\{N,2\}$	$\{N,3\}$	$\{N,N\}$
$\{1,1\}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$-\frac{1}{6}$
$\{1,2\}$	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{4}$
$\{1,3\}$		$\frac{1}{8}$		$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{2,1\}$	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{4}$
$\{2,2\}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{6}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{6}$
$\{2,3\}$	$\frac{1}{8}$			$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{3,1\}$		$\frac{1}{8}$		$-\frac{1}{8}$	$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{3,2\}$	$\frac{1}{8}$			$-\frac{1}{8}$	0	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$
$\{3,3\}$	$-\frac{1}{8}$	$-\frac{1}{8}$		$-\frac{1}{4}$	$-\frac{1}{24}$	$-\frac{1}{24}$	$\frac{1}{12}$	$-\frac{1}{6}$

Figure 10: Expected Payout Matrix For Player II

strategy for Player I has an expected payout of no more than $\$\frac{1}{8}$, while several strategies for Player II are expected non-losers, e.g., $\{2,2\}$, with four expected winning strategies: $\{*, N\}$. The expected payout of strategy $\{N, N\}$ for Player II is at most $\$\frac{1}{4}$, which is the same as is possible under, say, $\{1, N\}$, yet $\{N, N\}$ is the strategy for Player II that has six maximum payouts, where all other $\{*, N\}$ strategies have only three maximum payouts, against all strategies chosen by Player I.

3. Application To Risk Management

Risk management may be viewed as the utility function in a two-person zero-sum game between the "Public" and the "Menace." Other names for the "Menace" could be "state of nature," "natural forces," "entropy," "arbitrary acts," or "fate." However, "Menace" communicates the "risk" the process faces in the consumption of process services.

Whatever reduction there is in the risk of, say, property damage or personal injury due to the actions, choices, and strategies of the public, executed through its representatives, e.g., the industry, there is an equal reduction in the menace of engaging in process activities. The "Menace" acts to deteriorate the safe operation of process activities by wearing out parts, causing random defects to manifest, bringing together disparate coincidences to produce a singular event, inducing uncharacteristic process behavior (error), etc. The public acts through its representatives to improve the safe operation of process activities by performing inspections, issuing warnings and guidance, controlling and coordinating vital services, investigating and reporting on incidents, etc. These competing groups of actions may be modeled as a sequence of "moves," first by the "Menace" and then by the "Public." The public choices would be the options under a given circumstance, e.g., review documents, issue a directive, do nothing, etc.

The objective of each game would be to minimize the menace to the process, as measured by a utility function, for an individual process, a peer group set of processes, geographic-specific operations, or any other addressable unit of process activities, or combination of them.

The period for moves could be based on time units, such as "each day a random process makes a move, the Menace moves into a particular state, another random process makes another move, and the public counters with its own choices by the end of the day." The period could be in terms of events, such as a "initiate, infant-mortality, steady-state, wearout, retirement" cycle. In this case, the menace and random processes make moves interweaved with those of the public, before, during, and after each of these process situations. The public makes its moves in the uncertainty of the random processes, and in anticipation of the menace's actions, or in reaction to its perceived manifestations. The public does not see the menace acting in collusion with anyone or anything, so the conditions necessary for a two-person zero-sum game are confirmed.

Once modeled in this manner, industry operations could be guided by a strategy that optimizes, in some sense, risk as measured by the payout function. Even though individual industry personnel would not necessary be bound by rules established by the model, it could nevertheless be used to quantify in an objective manner the change in risk manifested by the chosen public moves.

It is possible to extend this analogy to more than two players. The "Menace" may be partitioned into more basic risk drivers, such as economically driven, technology related, and purely random primitives, each of which could be considered a player in a more complicated game. However, for the purposes of demonstrating in this document the usefulness and potential of a game theoretic approach to risk management, the two-person (the "Public" versus the "Menace") zero-sum game will be pursued.

4. Risk Management Games

Consider the case of an individual process under the supervision of a management team. The Public is represented by the actions of this team, and the Menace is represented by circumstances that make the individual process a risk to the Public. The *Risk Management Level 1 Game* is played as follows: At the beginning the risk s is at a given level s_0. Random processes occur first that combine to change the risk to a new level $s_0 + \rho_0$, where ρ_0 may be positive or negative. The Menace moves next (selects an action from its strategic choices) and thereby engenders another change to the risk level: $s_0 + \rho_0 + m_0$, where m_0 is necessarily positive. Under this model, random processes may increase or decrease the risk, however, the Menace always moves to increase the risk. Further random processes occur next that combine to change the risk to a new level: $s_0 + \rho_0 + m_0 + \lambda_0$, where λ_0 may be positive or negative. The Public moves next (selects an action from its strategic choices) and thereby engenders yet another change to the risk level: $s_0 + \rho_0 + m_0 + \lambda_0 - x_0$, where x_0 is necessarily nonnegative, i.e., the Public's actions never "do harm" to the risk. This is now the new risk level for the next round of choices. However, after the first round, the random processes do not affect the risk until the game restarts:

$$
\begin{aligned}
s_1 &= s_0 + \rho_0 + m_0 + \lambda_0 - x_0 \\
s_{i+1} &= s_i + m_i - x_i, \text{ for } i = 0, 1, 2, ... \\
s_i &= \text{risk level at start of Round } i \\
\rho_0 &= \text{change in risk level due to random processes before the Menace has first moved}
\end{aligned}
$$

$m_i =$ change in risk level due to strategic choice of the Menace in Round i
$\lambda_0 =$ change in risk level due to random processes after the Menace has first moved
$x_i =$ change in risk level due to strategic choice of the Public in Round i

This form of the game is necessarily simple to ensure the calculations do not become too cumbersome for this demonstration.

Suppose at the beginning of the game the risk level for the process is 10 on a scale of 0 to ∞. Suppose random processes before and after the Menace moves may change the risk level by $+2$ with probability $\frac{3}{4}$ and by -1 with probability $\frac{1}{4}$. Neither the Menace nor the Public knows any of the choices of the random processes. Suppose further that the Menace may choose to increase the risk by 1 to 2 on any move. Finally, suppose the Public may choose to perform actions that decrease the risk by 1 to 2 on any move. The sum of the Public's strategic choices is called the *outlay*, and the product of the outlay with the increase in risk level is called the *weighted outlay*. The object of the game is to minimize the weighted outlay after three rounds.

We have

$$h = \{\{2,2\}, \{2,-1\}, \{-1,2\}, \{-1,-1\}\}$$

which correspond to changes in the risk of $\{4, 1, -2\}$ with distribution function

$$P(\{2,2\}) = \frac{9}{16}, P(\{2,-1\}) = P(\{-1,2\}) = \frac{3}{16}, P(\{-1,-1\}) = \frac{1}{16}.$$

The strategic choices in each round for the Menace are $\{\{1\}, \{2\}\}$, and the strategic choices in each round for the Public are also $\{\{1\}, \{2\}\}$. The outcomes (the change to the risk level) are $r \in \mathcal{R} = \{-5, -4, ..., 6, 7\}$, and

$$F_1 = F_2 = \left\{ \begin{array}{c} \{1,1,1\}, \{1,1,2\}, \{1,2,1\}, \{1,2,2\}, \\ \{2,1,1\}, \{2,1,2\}, \{2,2,1\}, \{2,2,2\} \end{array} \right\}$$

For $f_1 = \{1,1,1\}$, the associated outcome probabilities are found in Figure 11. For

	r									
f_2	-5	-4	-3	-2	-1	0	1	2	3	4
$\{1,1,1\}$				$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$
$\{1,1,2\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,1,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{2,1,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,1\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,2\}$	$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$			

Figure 11: Outcome Probabilities For $f_1 = \{1,1,1\}$

$f_1 = \{\{1,1,2\}, \{1,2,1\}, \{2,1,1\}\}$, they are found in Figure 12.

	r									
f_2	-4	-3	-2	-1	0	1	2	3	4	5
$\{1,1,1\}$				$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$
$\{1,1,2\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,1,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{2,1,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,1\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,2\}$	$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$			

Figure 12: Outcome Probabilities For $f_1 = \{\{1,1,2\}, \{1,2,1\}, \{2,1,1\}\}$

For $f_1 = \{\{1,2,2\}, \{2,1,2\}, \{2,2,1\}\}$, they are found in Figure 13. And for $f_1 = \{2,2,2\}$, they are found in Figure 14.

	r									
f_2	-3	-2	-1	0	1	2	3	4	5	6
$\{1,1,1\}$				$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$
$\{1,1,2\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,1,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{2,1,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,1\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,2\}$	$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$			

Figure 13: Outcome Probabilities For $f_1 = \{\{1,2,2\},\{2,1,2\},\{2,1,1\}\}$

	r									
f_2	-2	-1	0	1	2	3	4	5	6	7
$\{1,1,1\}$				$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$
$\{1,1,2\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{1,2,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,1,1\}$			$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$	
$\{2,1,2\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,1\}$		$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$		
$\{2,2,2\}$	$\frac{1}{16}$			$\frac{3}{16}$			$\frac{9}{16}$			

Figure 14: Outcome Probabilities For $f_1 = \{2,2,2\}$

If the Menace is equally likely to choose any of its strategic choices at each round, then the outcome distribution against the Public strategies is given in Figure 15.

The expected outcome (expected change in risk level Δs) matrix, stated by Public strategy, with outlay and weighted outlay, is given in Figure 16.

The optimal strategy for the Public is $\{2,2,2\}$, since it has the lowest weighted outlay (6.5625). This shows that the maximum outlay (6) is justified, since it is the most powerful strategy for minimizing the risk level while taking outlay into consideration. Contrary to intuition, however, the least optimal strategy for the Public is *not* $\{1,1,1\}$. While it does produce the highest increase in the risk level of all Public strategies (3.53125), those strategies with outlay of 4 are worse when the risk level *and* outlay are considered.

These results could change drastically if any of the conventions, e.g., the uniform distribution over the Menace strategic choices, or the calculation of the outlay function, is amended.

5. Reduced Games

As the examples have demonstrated, the required calculations to evaluate multi-step games (even with a few players) quickly becomes highly intricate. A reduced form of a game is need to control complexity. Consider the following additional definitions.

1. A two-person zero-sum power game $\mathcal{G}_0' = (F_1', F_2', h', M')$ is a **reduction** of another two-person zero-sum power game $\mathcal{G}_0 = (F_1, F_2, h, M)$, written $\mathcal{G}_0' \veebar \mathcal{G}_0$, if

   (a) $F_1' = F_1$, and $\exists \xi : F_2 \overset{\text{onto}}{\to} F_2'$ such that $M(f_1, f_2, h) = M'(f_1, \xi(f_2)), \forall f_1 \in F_1, f_2 \in F_2$; *OR*

   (b) $F_2' = F_2$, and $\exists \zeta : F_1 \overset{\text{onto}}{\to} F_1'$ such that $M(f_1, f_2, h) = M'(\zeta(f_1), f_2), \forall f_1 \in F_1, f_2 \in F_2$.
   If ξ or ζ is a bijection, then $\mathcal{G}_0'$ is obtained from $\mathcal{G}_0$ by relabeling, otherwise $\mathcal{G}_0'$ is obtained from $\mathcal{G}_0$ by elimination of duplicates and relabeling.

2. $\mathcal{G}_0 = (F_1, F_2, h, M)$ and $\mathcal{G}_0' = (F_1', F_2', h', M')$ are **equivalent** two-person zero-sum power games, written $\mathcal{G}_0 \sim \mathcal{G}_0'$, if there is a finite sequence $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$ with $\mathcal{G}_0' = \mathcal{G}_n$, and for each $i = 1, 2, ..., n$, either $\mathcal{G}_{i-1} \veebar \mathcal{G}_i$ or $\mathcal{G}_i \veebar \mathcal{G}_{i-1}$.
3. A two-person zero-sum power game $\mathcal{G}_0 = (F_1, F_2, h, M)$ is called **finite** if F_1 and F_2 are finite, i.e., $F_i = \left\{ f_i^{(1)}, f_i^{(2)}, ..., f_i^{(|F_i| < \infty)} \right\}$ for $i = 1, 2$.

	r												
f_2	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$\{1,1,1\}$				$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$
$\{1,1,2\}$			$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$	
$\{1,2,1\}$			$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$	
$\{1,2,2\}$		$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$		
$\{2,1,1\}$			$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$	
$\{2,1,2\}$		$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$		
$\{2,2,1\}$		$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$		
$\{2,2,2\}$	$\frac{1}{128}$	$\frac{3}{128}$	$\frac{3}{128}$	$\frac{4}{128}$	$\frac{9}{128}$	$\frac{9}{128}$	$\frac{12}{128}$	$\frac{27}{128}$	$\frac{27}{128}$	$\frac{9}{128}$			

Figure 15: Outcome Distribution Against The Public Strategies

f_2	Outlay	Δs	Weighted Outlay
$\{1,1,1\}$	3	3.53125	10.59375
$\{1,1,2\}$	4	2.71875	10.875
$\{1,2,1\}$	4	2.71875	10.875
$\{1,2,2\}$	5	1.90625	9.53125
$\{2,1,1\}$	4	2.71875	10.875
$\{2,1,2\}$	5	1.90625	9.53125
$\{2,2,1\}$	5	1.90625	9.53125
$\{2,2,2\}$	6	1.09375	6.5625

Figure 16: Expected Outcome Matrix By Public Strategy

4. Two strategies $f^{(1)}$ and $f^{(2)}$ in a two-person zero-sum power game $\mathcal{G}_0 = (F_1, F_2, h, M)$ are said to be **duplicates** if $M\left(f^{(1)}, f_2\right) \equiv M\left(f^{(2)}, f_2\right)$ identically in $f_2 \in F_2$.
5. The **matrix** of a two-person zero-sum power game $\mathcal{G}_0 = (F_1, F_2, h, M)$ is the $|F_1| \times |F_2|$ matrix with $(i,j)^{th}$ entries of $M\left(f_1^{(i)}, f_2^{(j)}\right)$.
6. A two-person zero-sum power game $\mathcal{G}_0 = (F_1, F_2, h, M)$ is **completely reduced** if

   (a) $f_1^{(i)} \neq f_1^{(j)} \Longrightarrow M\left(f_1^{(i)}, f_2\right) \neq M\left(f_1^{(j)}, f_2\right)$, for some $f_2 \in F_2$, *AND*

   (b) $f_2^{(i)} \neq f_2^{(j)} \Longrightarrow M\left(f_1, f_2^{(i)}\right) \neq M\left(f_1, f_2^{(j)}\right)$, for some $f_1 \in F_1$.

The terms used in these definitions are meant to be intuitive. For example, a completely reduced game is one in which no player has an available duplicate strategy.

Each of these definitions has a canonical extension to k-player games. For example, there would be k-many possible functions in the definition of a reduction of a k-player game, and the matrix would be a k-dimensional hypermatrix of cardinality $\prod_{n=1}^{k} |F_n|$. However, for purposes of discussion at this point, further points will be discussed strictly in terms of two-person zero-sum games.

6. Sufficiency Of Reduced Games

There is no loss of information in the use of a reduced game in place of its original form.

Lemma 1 *Every two-person zero-sum power game is equivalent to itself:* $\mathcal{G}_0 \sim \mathcal{G}_0$.

Proof. Let $n = 0$ in the definition of the equivalence of two-person zero-sum power games. ■

Lemma 2 *If* $\mathcal{G}_0 \sim \mathcal{G}'_0$, *then* $\mathcal{G}'_0 \sim \mathcal{G}_0$.

Proof. If there is a finite sequence $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$ with $\mathcal{G}'_0 = \mathcal{G}_n$, and, without loss of generality, for each $i = 1, 2, ..., n$, $\mathcal{G}_{i-1} \veebar \mathcal{G}_i$, then there is a finite sequence $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_n$ with $\mathcal{G}'_0 = \mathcal{G}_n$, and for each $i = 1, 2, ..., n$, $\mathcal{G}_i \veebar \mathcal{G}_{i-1}$. ■

Lemma 3 *If $\mathcal{G}_0 \sim \mathcal{G}_0'$ and $\mathcal{G}_0' \sim \mathcal{G}_0''$, then $\mathcal{G}_0 \sim \mathcal{G}_0''$.*

Proof. If $\mathcal{G}_0 \sim \mathcal{G}_0'$, then there is a finite sequence $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_{n_0}$ with $\mathcal{G}_0' = \mathcal{G}_{n_0}$, such that, without loss of generality, for each $i = 1, 2, ..., n_0$, $\mathcal{G}_{i-1} \veebar \mathcal{G}_i$. Similarly, if $\mathcal{G}_0' \sim \mathcal{G}_0''$, then there is a finite sequence $\mathcal{G}_{n_0+1}, \mathcal{G}_{n_0+2}, ..., \mathcal{G}_{n_0+n_1}$ with $\mathcal{G}_0'' = \mathcal{G}_{n_0+n_1}$, such that, without loss of generality, for each $j = n_0+1, n_0+2, ..., n_0+n_1$, $\mathcal{G}_{j-1} \veebar \mathcal{G}_j$. Hence, there is a finite sequence $\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_{n_0+n_1}$ with $\mathcal{G}_0'' = \mathcal{G}_{n_0+n_1}$, such that for each $k = 1, 2, ..., n_0+n_1$, $\mathcal{G}_{k-1} \veebar \mathcal{G}_k$. This means $\mathcal{G}_0 \sim \mathcal{G}_0''$. ■

Lemmas 1, 2, and 3 prove that "$\sim$" is an *equivalence relation* among the class of two-person zero-sum games. Two two-person zero-sum games $\mathcal{G}_0$ and $\mathcal{G}_0'$ are in the same *equivalence class* if $\mathcal{G}_0 \sim \mathcal{G}_0'$.

Lemma 4 *If $F_1' \subset F_1$, a sufficient condition that $\mathcal{G}_0' = (F_1', F_2, h, M')$, with $M'(f_1', f_2) = M(f_1', f_2), \forall f_1' \in F_1', f_2 \in F_2$, is equivalent to $\mathcal{G}_0 = (F_1, F_2, h, M)$ is that $\forall f_1 \in F_1, \exists f_1' \in F_1'$ such that $M(f_1', f_2) \equiv M(f_1, f_2)$ identically in $f_2 \in F_2$.*

Proof. Let $A = \{\forall f_1 \in F_1, \exists f_1' \in F_1', \text{ such that } M(f_1', f_2) = M(f_1, f_2), \forall f_2 \in F_2\}$. It suffices to show $A \Longrightarrow \{\mathcal{G}_0' \veebar \mathcal{G}_0\}$.

Suppose A holds, and that $\mathcal{G}_0' = (F_1', F_2, h, M')$, with $M'(f_1', f_2) = M(f_1', f_2), \forall f_1' \in F_1', f_2 \in F_2$. Since F_2 is common to $\mathcal{G}_0'$ and $\mathcal{G}_0$, all references to f_2 are to elements of the common F_2. Let $k : F_1 \overset{\text{onto}}{\to} F_1'$ be defined by

$$k(f) = \begin{cases} f, & f \in F_1' \\ h(f), & f \in F_1 - F_1' \end{cases}$$

where $h : F_1 \to F_1'$ is the function defined by A. Note that k is *onto* F_1', and only F_1', since h maps into F_1', and $k(F_1') = F_1'$.

For $f_1' \in F_1'$, we have $M(f_1', f_2) = M'(f_1', f_2), \forall f_2$ (by supposition on $\mathcal{G}_0'$), and for $f_1 \in F_1 - F_1'$, we have $M(f_1, f_2) = M(h(f_1), f_2), \forall f_2$, by the definition of h, and $M(h(f_1), f_2) = M'(h(f_1), f_2), \forall f_2$, since $h(f_1) \in F_1'$. So $\forall f_1 \in F_1$, we have $M(f_1, f_2) = M'(k(f_1), f_2), \forall f_2$, which means $\mathcal{G}_0' \veebar \mathcal{G}_0$. ■

Theorem 5 *Every two-person zero-sum power game is equivalent to a completely reduced game.*

Proof. Let $\mathcal{G}_0 = (F_1, F_2, h, M)$ be a two-person zero-sum power game. Let F_1' be the set of all elements of F_1 with duplicates removed, and let F_1'' be the set of those duplicates so removed. This partitions F_1 into two sets, though not uniquely (since different duplicates may be "left behind" in F_1'), however disjointedly: $F_1' \cup F_1'' = F_1$, and $F_1' \cap F_1'' = \emptyset$. Let $\mathcal{G}_0' = (F_1', F_2, h, M')$ be the two-person zero-sum power game derived from F_1', i.e., $M'(f_1, f_2) = M(f_1, f_2)$ when $f_1 \in F_1', \forall f_2 \in F_2$.

Note how $f_1'' \in F_1''$ means there is an $f_1' \in F_1'$ such that $M(f_1'', f_2) = M(f_1', f_2), \forall f_2 \in F_2$, since simply being in F_2'' means there is a duplicate for it in F_1'. By Lemma 4, we have $\mathcal{G}_0' \sim \mathcal{G}_0$.

Since the same arguments apply in Lemma 4 and the statements above for F_2 replacing F_1, then for $\mathcal{G}_0'' = (F_1', F_2', h, M'')$, where $M''(f_1', f_2) = M'(f_1', f_2)$ when $f_2 \in F_2, \forall f_1' \in F_1'$, and $f_2'' \in F_2''$ means there is an $f_2' \in F_2'$ such that $M(f_1', f_2'') = M(f_1', f_2'), \forall f_1' \in F_1'$, we have $\mathcal{G}_0'' \sim \mathcal{G}_0'$. By Lemma 3, $\mathcal{G}_0'' \sim \mathcal{G}_0$.

Furthermore, if $f_1^{(1)}, f_1^{(2)} \in F_1'$and $M''\left(f_1^{(1)}, f_2'\right) = M''\left(f_1^{(2)}, f_2'\right), \forall f_2' \in F_2'$, then $f_1^{(1)}$ and $f_1^{(2)}$ would be duplicates in F_1', which would contradict the definition of F_1'. Therefore,

$$f_1^{(1)} \neq f_1^{(2)} \Longrightarrow M''\left(f_1^{(1)}, f_2'\right) \neq M''\left(f_1^{(2)}, f_2'\right) \text{ for some } f_2' \in F_2'.$$

Likewise, if $f_2^{(1)}, f_2^{(2)} \in F_2'$and $M''\left(f_1', f_2^{(1)}\right) = M''\left(f_1', f_2^{(2)}\right), \forall f_1' \in F_1'$, then $f_2^{(1)}$ and $f_2^{(2)}$ would be duplicates in F_2', which would contradict the definition of F_2'. Therefore,

$$f_2^{(1)} \neq f_2^{(2)} \Longrightarrow M''\left(f_1', f_2^{(1)}\right) \neq M''\left(f_1', f_2^{(2)}\right) \text{ for some } f_1' \in F_1'.$$

Hence, $\mathcal{G}_0'' = (F_1', F_2', h, M'')$ is completely reduced, and $\mathcal{G}_0'' \sim \mathcal{G}_0$. ■

Theorem 6 *Two finite two-person zero-sum power games $\mathcal{G}_A$ and $\mathcal{G}_B$ with matrices A and B, respectively, are equivalent if B is obtained from A by a permutation of rows or columns.*

Proof. If $\mathcal{G}_A = \left(F_1^{(\mathcal{G}_A)}, F_2^{(\mathcal{G}_A)}, h_{\mathcal{G}_A}, M_{\mathcal{G}_A}\right)$ has matrix A, then $A = \left\{M_{\mathcal{G}_A}\left(f_{\mathcal{G}_A}^{(i)}, f_{\mathcal{G}_A}^{(j)}\right)\right\}$, where $M_{\mathcal{G}_A}$ is the payout function for $\mathcal{G}_A$. Similar definitions hold for

$$\mathcal{G}_B = \left(F_1^{(\mathcal{G}_B)}, F_2^{(\mathcal{G}_B)}, h_{\mathcal{G}_B}, M_{\mathcal{G}_B}\right)$$

and

$$B = \left\{M_{\mathcal{G}_B}\left(f_{\mathcal{G}_B}^{(i)}, f_{\mathcal{G}_B}^{(j)}\right)\right\}$$

A permutation of rows of the matrix A is a function:

$$r : \left\{1, 2, ..., \left|F_1^{(\mathcal{G}_A)}\right| < \infty\right\} \overset{\text{1-1, onto}}{\to} \left\{1, 2, ..., \left|F_1^{(\mathcal{G}_A)}\right| < \infty\right\}$$

. Now suppose B is obtained from A by a permutation of rows, i.e., by applying r to the rows of A. This means

$$M_{\mathcal{G}_B}\left(f_{\mathcal{G}_B}^{(i)}, f_{\mathcal{G}_B}^{(j)}\right) = M_{\mathcal{G}_A}\left(f_{\mathcal{G}_A}^{(r(i))}, f_{\mathcal{G}_B}^{(j)}\right), \forall f_{\mathcal{G}_B}^{(j)}$$

Hence, $\mathcal{G}_A \veebar \mathcal{G}_B$, or $\mathcal{G}_A \sim \mathcal{G}_B$.
A permutation of columns of the matrix A is a function:

$$c : \left\{1, 2, ..., \left|F_2^{(\mathcal{G}_A)}\right| < \infty\right\} \overset{\text{1-1, onto}}{\to} \left\{1, 2, ..., \left|F_2^{(\mathcal{G}_A)}\right| < \infty\right\}$$

. Now suppose B is obtained from A by a permutation of columns, i.e., by applying c to the columns of A. This means

$$M_{\mathcal{G}_B}\left(f_{\mathcal{G}_B}^{(i)}, f_{\mathcal{G}_B}^{(j)}\right) = M_{\mathcal{G}_A}\left(f_{\mathcal{G}_B}^{(i)}, f_{\mathcal{G}_A}^{(c(j))}\right), \forall f_{\mathcal{G}_B}^{(i)}$$

Hence, $\mathcal{G}_A \veebar \mathcal{G}_B$, or $\mathcal{G}_A \sim \mathcal{G}_B$. ■

Theorems 5 and 6 further explain the statement made in the definition of reduction: "If ξ or ζ is a bijection (r and c in Theorem 6), then $\mathcal{G}_0'$ is obtained from $\mathcal{G}_0$ by relabeling (i.e., permutations), otherwise $\mathcal{G}_0'$ is obtained from $\mathcal{G}_0$ by elimination of duplicates (see the proof of Theorem 5) and relabeling."

These theorems also demonstrate that it suffices to optimize (in some sense) the matrix of a completely reduced two-person zero-sum game to address the same optimization on all such equivalent games. As suggested in Theorem 6, the matrix optimization should be invariant under row and column permutations (such as maximizing the determinate of the matrix square).

For example, in the Chance Element Game given before, (without units) the 2×8 payout matrix was

$$\begin{pmatrix} -3.60 & +0.30 & -3.60 & +0.30 & +0.30 & +4.20 & +0.30 & +4.20 \\ +4.20 & -0.30 & -0.30 & -4.80 & +4.20 & -0.30 & -0.30 & -4.80 \end{pmatrix}.$$

Note how this shows the game is not completely reduced, as the second and seventh columns are the same. This means strategies $\{3, 3, 4\}$ and $\{4, 4, 3\}$ are duplicate strategies for Player II under the payout function defined in that example. The completely reduced game would have a payout function such as this:

$$\begin{pmatrix} -3.60 & +0.30 & -3.60 & +0.30 & +0.30 & +4.20 & +4.20 \\ +4.20 & -0.30 & -0.30 & -4.80 & +4.20 & -0.30 & -4.80 \end{pmatrix}$$

Any two-person zero-sum game whose payout matrix is

$$\begin{pmatrix} +4.20 & -0.30 & -0.30 & -4.80 & +4.20 & -0.30 & -4.80 \\ -3.60 & +4.20 & -3.60 & +0.30 & +0.30 & +0.30 & +4.20 \end{pmatrix}$$

would be equivalent to the Chance Element Game.

Similarly for the Stick or Change Game (in its original form), since each row and column of the expected payout matrix is unique, it is a completely reduced game.

The Risk Management Level 1 Game is not completely reduced, as any two strategies with equal outlay result in the same Δs.

7. Summary And Future Development

We have now seen that a two-person zero-sum power game may model the interactions between process services and the public representatives who strive to make those services as minimally risky as possible. Any completely reduced game is representative of all other games in the same equivalence class of similarly defined games. This simplifies the structure of most risk games into one of several prototype game classes, depending on the nature of process services involved. The public may make use of such models to allocate resources, such as was measured in the outlay of the Risk Management Level 1 Game, in as effective and efficient manner as possible, allowing for the deterministic and random variations of risk-related factors beyond the control or influence of the public.

Additional concepts may be developed to simplify the calculations involved in risk games. Matrix algebra techniques would simplify many questions regarding the optimal nature of unidimensional utility functions. Linear programming techniques would be involved in the simultaneous maximization of multiple utility functions. Presentation graphics and parametrized "what if" calculators could be developed for particular circumstances, such as for ad-hoc peer groups. Useful theory concepts include:

1. *Imperfect Spying* - This is the circumstance when one player knows some, but not necessarily all strategic choices for the other player's next round, and makes strategic choices based on that knowledge in the current round.
2. *Incomplete Information* - The circumstance where the information sets do not reflect the totality of the information potentially available to a player.
3. *Game Value* - the strategy that necessarily leads to the optimal utility, or arbitrarily close to it, regardless of other players' chosen strategies, is called the **value** of a game; knowing this value could be used to guide strategic choices in critical rounds.
4. *Pure versus Mixed Strategies* - A **pure** strategy is a strategy chosen from F; a **mixed** strategy is a hybrid of all strategies in F defined by a probability distribution that a player used to make strategic choices.
5. *Suboptimization* - Optimal utility may have multiple dimensions; optimization in one dimension versus global optimization may be of use in particular circumstances.

These concepts, and many more, may be developed in the context of assisting the public's representatives in their pursuit of optimal risk management.