The Epsilon Skew Gamma Distribution ESΓ

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Abstract
In this paper, we introduce a new skewed distribution family, the Epsilon Skew Gamma (ESΓ). The ESΓ contains the reflected Γ distribution as a special case. The ESΓ has four parameters the location, scale, shape, and skewness. We derive the pdf and distribution functions of the ESΓ, basic properties, the first four moments, and the moment generating function and the characteristic function. We also derive the maximum likelihood estimators (MLE) for the parameters and examine their asymptotic variance, and calculate Fisher information matrix for the ESΓ.

Key Words: Epsilon Skew distributions, Reflected gamma, Skewed distributions, Fisher information Matrix.

1. Introduction

In real life applications, we seek to have distributions for analyzing skewed data and involving tail behavior. In this century, we have seen a good attention for fitting data using asymmetric distributions in order to represent the variations in the cases study and modeling data that contain outliers from both sides of the distribution. Many efforts have been motivated to introduce skew-symmetric distributions which can account for both skewness and kurtosis, see e.g., Johns and Faddy (2003), Azzalini, et al. (2003), Arellano-Valle et al. (2005), Gupta et al. (2002), Elsalloukh (2004, 2005, and 2008) for Epsilon Skew Exponential Power (ESEP), and Epsilon Skew Laplace (ESL) distributions, Arnold and Beaver (2000), Wahed and Ali (2001), and Nadarajah and Kotz (2003). In this research, we include definitions and basic properties, the MLE and MME estimation of the parameters, moment generating and characteristic functions, and Fisher information matrix of the ESΓ distribution.

2. Definition and Basic Properties of the ESΓ Distribution

Borghi (1965) defined the pdf of the reflected Γ distribution as

$$f(x;\theta,\beta,k) = \frac{1}{2\Gamma(k)\beta^k}|(x-\theta)|^{k-1}\, e^{-\frac{|x-\theta|}{\beta}} \qquad x \in R\,, \tag{1}$$

where $\theta \in R$, β and $k > 0$ are the location, scale, and shape parameters, respectively. This distribution is symmetric about the location parameter and has a heavier or lighter tails than the normal distribution depending on the value of the shape parameter. Figure 1 shows the reflected Γ distribution with different values of the shape parameter k. The standard form of this distribution, when $\theta = 0$ and $\beta = 1$, is

$$f(x;k) = \frac{1}{2\Gamma(k)}|x|^{(k-1)}\, e^{-|x|}\,.$$

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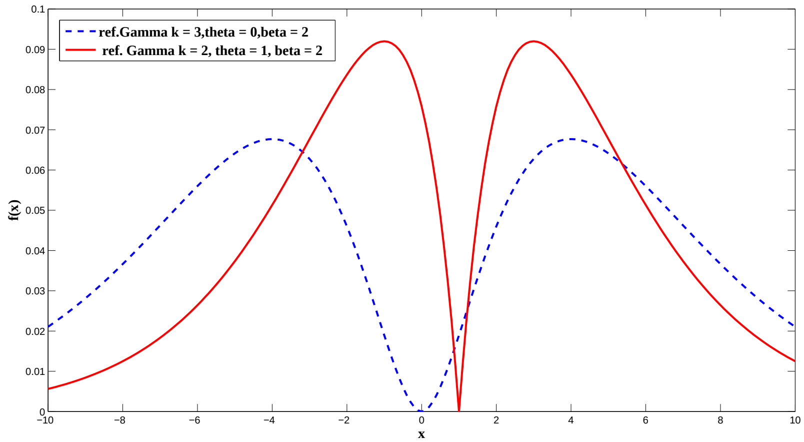

Figure 1: Reflected Γ Density Functions

The linear estimation of θ and β for a given k requires the moments of order statistics of samples which are drawn from this distribution Kantam (1991). In order to handle distributions that can deal with both the skewness and heavier or lighter tails than the normal distribution, we can extend the reflected Γ distribution (1) by adding a new parameter, ϵ, which accommodates for the skewness and allows to convert the distribution from symmetric to asymmetric, where the mean, mode, and median of the distribution occur at different points and the distribution has the ability to detect outliers from the left and right sides. When $\epsilon < 0$, the distribution has a longer tail on the left side than the right side and when $\epsilon > 0$, the right tail is longer than the left. Note that when $\epsilon = 0$, the distribution is symmetric.

proposition 1. *If $Z \sim$ ESEP $(\theta, \beta, k, \epsilon)$ is a random variable, then the random variable $X = \left(\frac{(Z-\theta)}{\epsilon_i \beta}\right)^k$ is $X \sim$ ESΓ $(0, 1, 1/k, \epsilon_i)$. This is a transformation case with standardized form, where $i = 1, 2$, $\epsilon_1 = 1/(1-\epsilon)$ for $x \geq 0$, and $\epsilon_2 = 1/(1+\epsilon)$ for $x < 0$.*

Proof. The pdf of the Epsilon Skew Exponential Power distribution ESEP, Elsalloukh (2004), of the random variable Z with the parameters $(\theta, \beta, k, \epsilon)$ is

$$f(z) = \frac{k}{2\beta\Gamma(1/k)} \begin{cases} e^{-\frac{|z-\theta|^k}{(\beta(1-\epsilon))^k}} & \text{if } z \geq \theta \\ e^{-\frac{|z-\theta|^k}{(\beta(1+\epsilon))^k}} & \text{if } z < \theta \ . \end{cases} \tag{2}$$

For the first orthant $z \geq \theta$, let

$$x = \left(\frac{z-\theta}{(1-\epsilon)\beta}\right)^k$$

then

$$\frac{dz}{dx} = \frac{1}{k}(1-\epsilon)\beta x^{(\frac{1}{k}-1)} \, .$$

Likewise, for $z < \theta$, we have $x = \left(\frac{z-\theta}{(1+\epsilon)\beta}\right)^k$ and $\frac{dz}{dx} = \frac{1}{k}(1+\epsilon)\beta x^{(\frac{1}{k}-1)}$, by substituting

z into (2) for both cases, we have

$$f(x) = \frac{k}{2\beta\Gamma(1/k)} \begin{cases} \frac{1}{k}\, x^{(\frac{1}{k}-1)}(1-\epsilon)\beta\, e^{-x} & \text{if } x \geq 0 \\ \frac{1}{k}\, x^{(\frac{1}{k}-1)}(1+\epsilon)\beta\, e^{-x} & \text{if } x < 0\ . \end{cases}$$

Therefore, one can obtain the pdf of standard ESΓ distribution as

$$f(x) = \frac{1}{2\Gamma(1/k)} \begin{cases} (1-\epsilon)x^{(\frac{1}{k}-1)}\, e^{-x} & \text{if } x \geq 0 \\ (1+\epsilon)x^{(\frac{1}{k}-1)}\, e^{-x} & \text{if } x < 0\ . \end{cases}$$

□

Definition 1. *A random variable X is said to have an ESΓ distribution with parameters $\theta \in R$, $\beta > 0$, $k > 0$, and $|\epsilon| < 1$ that are location, scale, shape, and skewness parameters, respectively, if it has the pdf*

$$f(x) = \frac{1}{2\Gamma(k)\beta^k} \begin{cases} \left(\frac{(x-\theta)}{(1-\epsilon)}\right)^{(k-1)} e^{-\frac{(x-\theta)}{\beta(1-\epsilon)}} & \textit{if } x \geq \theta \\ \left(\frac{(\theta-x)}{(1+\epsilon)}\right)^{(k-1)} e^{-\frac{(\theta-x)}{\beta(1+\epsilon)}} & \textit{if } x < \theta\ . \end{cases} \tag{3}$$

Note that when $\epsilon = 0$, $X \sim$ symmetric reflected Γ (θ, β, k) distribution, when $\epsilon > 0$, the distribution is skewed to the right and the right tail is longer than the left tail, and when $\epsilon < 0$, the distribution is skewed to the left and the left tail is longer than the right tail.

proposition 2. *If $X \sim$ ES$\Gamma(\theta, \beta, k, \epsilon)$, then the cumulative distribution, $F(x)$, function of X is*

$$F(x) = \begin{cases} 1 - \frac{(1-\epsilon)}{2\Gamma(k)}\Gamma(k, g(x)) & for\, x \geq \theta \\ \frac{(1+\epsilon)}{2\Gamma(k)}\Gamma(k, h(x)) & for\, x < \theta\ . \end{cases}$$

Proof. for $X \geq \theta$

$$\begin{aligned} p(X \leq x) &= \int_\theta^x \frac{1}{2\Gamma(k)\beta^k} \left(\frac{(y-\theta)}{(1-\epsilon)}\right)^{k-1} e^{-\frac{(y-\theta)}{\beta(1-\epsilon)}} dy \\ &= 1 - \frac{1}{2\Gamma(k)\beta^{k-1}} \int_x^\infty \left(\frac{(y-\theta)}{(1-\epsilon)}\right)^{k-1} e^{-\frac{(y-\theta)}{\beta(1-\epsilon)}} dy\ . \end{aligned} \tag{4}$$

Let

$$z = \frac{y-\theta}{\beta(1-\epsilon)}\,,$$

then (4) becomes

$$\begin{aligned} p(X \leq x) &= 1 - \frac{1}{2\Gamma(k)\beta^k} \int_{g(x)}^\infty \left(\frac{z\beta(1-\epsilon)+\theta-\theta}{(1-\epsilon)}\right)^{k-1} \beta(1-\epsilon)\, e^{-z}\, dz \\ &= 1 - \frac{(1-\epsilon)}{2\Gamma(k)}\Gamma(k, g(x))\,, \end{aligned}$$

where

$$g(x) = \frac{(x-\theta)}{\beta(1-\epsilon)}$$

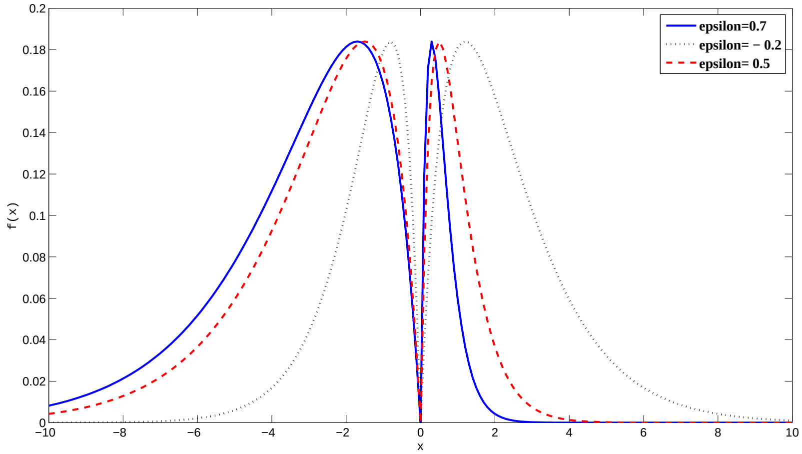

Figure 2: ESΓ Density Functions for Different Values for ϵ

and $\Gamma(k, a(x)) = \int_{a(x)}^{\infty} z^{k-1}e^{-z}dz$ is the upper incomplete Γ function. Similarly, for $X < \theta$, let

$$z = \frac{\theta - y}{\beta(1+\epsilon)},$$

then

$$\begin{aligned} p(X \leq x) &= \frac{1}{2\Gamma(k)\beta^k} \int_{\infty}^{h(x)} \left(\frac{\theta - \theta + z\beta(1+\epsilon)}{(1+\epsilon)}\right)^{k-1} \left(-\beta(1+\epsilon)\right) e^{-z}\, dz \\ &= \frac{(1+\epsilon)}{2\Gamma(k)} \Gamma(k, h(x)), \end{aligned}$$

where

$$h(x) = \frac{(\theta - x)}{\beta(1+\epsilon)}$$

□

Figure 2 shows ESΓ with different values for ϵ, Figure 3 shows the cdf of ESΓ with $\epsilon = 0.3$, and Figure 4 shows the difference between ESΓ and reflected Γ distributions. Note that when $k = 1$, (3) becomes ESL defined in Elsalloukh (2005), (2008), and Almousawi (2011).

3. Central Moments and First Four Moments for the ESΓ Distribution

This section is devoted to derive the central moments and first four moments of ESΓ distribution by using the following proposition.

proposition 3. *If $X \sim ES\Gamma(\theta, \beta, k, \epsilon)$, then the central moments, mean, variance, and skewness and kurtosis coefficients are, respectively*

1.

$$E(X-\theta)^n = \frac{\beta^n \Gamma(n+k)}{2\Gamma(k)} \left[(-1)^n (1+\epsilon)^{n+1} + (1-\epsilon)^{n+1}\right], \tag{5}$$

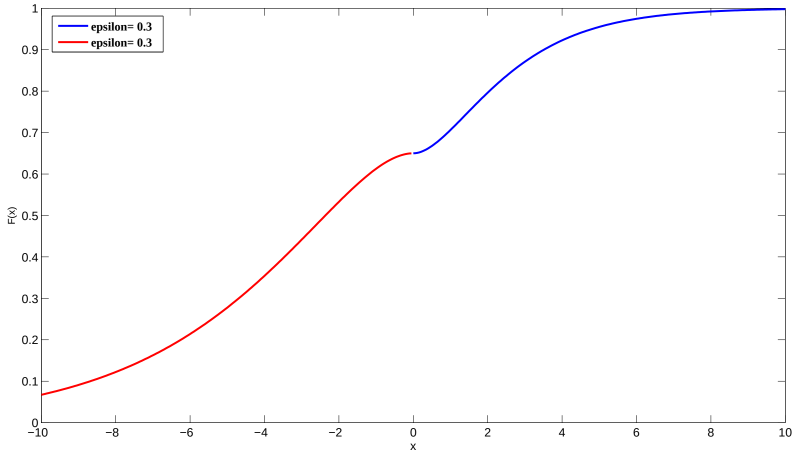

Figure 3: CDF For ESΓ Density Functions with ϵ= 0.3

2.
$$E(X) = \theta - 2k\beta\epsilon\,, \tag{6}$$

3.
$$Var(X) = \beta^2 k\left[(k+1)(1+3\epsilon^2) - 4k\epsilon^2\right]\,, \tag{7}$$

4.
$$\lambda_1 = \left\{\frac{1}{\Gamma(k)}\left[\Gamma(k+2) + (3\Gamma(k+2) - 4k^2\Gamma(k))\epsilon^2\right]\right\}^{-3/2} \left[\frac{-2k\epsilon\Gamma(k) + (1+3\epsilon^2)\Gamma(k+2) - 4\epsilon(1+\epsilon^2)\Gamma(k+3)}{\Gamma(k)}\right]\,, \tag{8}$$

5.
$$\lambda_2 = \left\{\frac{1}{\Gamma(k)}\left[\Gamma(k+2) + (3\Gamma(k+2) - 4k^2\Gamma(k))\epsilon^2\right]\right\}^{-2} \left[\frac{-2k\epsilon\Gamma(k) + (1+3\epsilon^2)\Gamma(k+2) - 4\epsilon(1+\epsilon^2)\Gamma(k+3) + 2\Gamma(k+4)(1+10\epsilon^2+5\epsilon^4)}{\Gamma(k)}\right]\,. \tag{9}$$

Proof. 1. If $n > 0$ and integer, then

$$\begin{aligned} E(X-\theta)^n &= \int_{-\infty}^{\infty}(x-\theta)^n f_{ES\Gamma}(x)\,dx \\ &= \frac{1}{2\Gamma(k)\beta^k}\int_{-\infty}^{\theta}(-1)^n(\theta-x)^n\left(\frac{\theta-x}{(1+\epsilon)}\right)^{k-1} e^{-\frac{(\theta-x)}{\beta(1+\epsilon)}}\,dx \\ &+ \frac{1}{2\Gamma(k)\beta^k}\int_{\theta}^{\infty}(x-\theta)^n\left(\frac{x-\theta}{(1-\epsilon)}\right)^{k-1} e^{-\frac{(x-\theta)}{\beta(1-\epsilon)}}\,dx\,. \end{aligned} \tag{10}$$

Let

$$z = \frac{\theta - x}{\beta(1+\epsilon)}$$

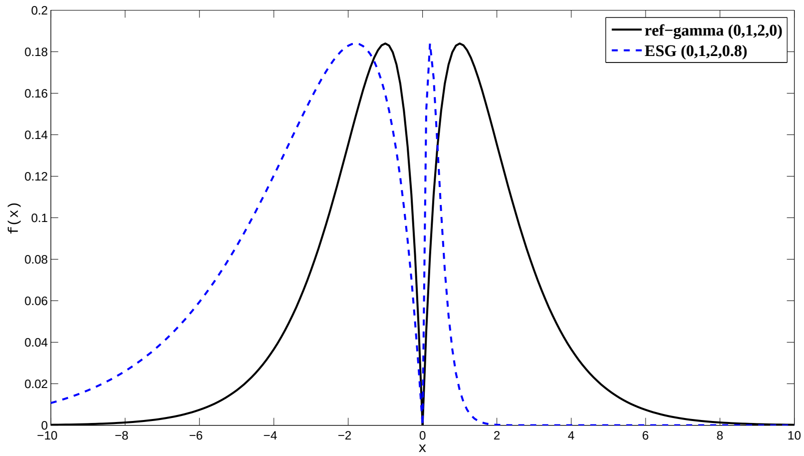

Figure 4: ESΓ and Reflected Γ Density Functions

for $X < \theta$ and

$$z = \frac{x-\theta}{\beta(1-\epsilon)}$$

for $X \geq \theta$, then

$$\begin{aligned} E(X-\theta)^n &= \frac{(-1)^n}{2\Gamma(k)\beta^k(1+\epsilon)^{k-1}} \int_\infty^0 (z\beta(1+\epsilon))^{n+k-1} \left(-\beta(1+\epsilon)\right) e^{-z}\, dz \\ &+ \frac{1}{2\Gamma(k)\beta^k(1-\epsilon)^{k-1}} \int_\infty^0 (z\beta(1-\epsilon))^{n+k-1} \left(-\beta(1-\epsilon)\right) e^{-z}\, dz \\ &= \frac{(-1)^n\beta^n(1+\epsilon)^{n+1}}{2\Gamma(k)}\Gamma(n+k) + \frac{\beta^n(1-\epsilon)^{n+1}}{2\Gamma(k)}\Gamma(n+k)\ . \end{aligned}$$

2. Let $\theta = 0$ and $n = 1$, then using (5) we have (6)
3. Since

$$Var(X) = E(X^2) - [E(X)]^2\ ,$$

letting $\theta = 0$ and $n = 2$, using part (2) and (5) we have (7)
4. Since

$$\lambda_1 = \frac{E(X-E(X))^3}{(Var(X)^2)^{3/2}}\ , \tag{11}$$

using (5) and (6) when $\theta = 0$ and $\beta = 1$, we have

$$E(X) = -2k\epsilon, \qquad E(X^2) = \frac{(1+3\epsilon^2)\Gamma(k+2)}{\Gamma(k)}\ ,$$

and

$$E(X^3) = \frac{-4\epsilon(1+\epsilon^2)\Gamma(k+3)}{\Gamma(k)}\ .$$

Thus

$$E(X-E(X))^3 = \frac{-2k\epsilon\Gamma(k) + (1+3\epsilon^2)\Gamma(k+2) - 4\epsilon(1+\epsilon^2)\Gamma(k+3)}{\Gamma(k)}\ , \tag{12}$$

substitute (12) and (7) in (11) we have (8)
5. Since

$$\lambda_2 = \frac{E(X - E(X))^4}{(Var(X))^2}\,, \tag{13}$$

using (5), when $\theta = 0$ and $\beta = 1$, we have

$$E(X^4) = \frac{2\Gamma(k+4)}{\Gamma(k)}(1 + 10\epsilon^2 + 5\epsilon^4)\,.$$

Thus

$$E[X - E(X)]^4 = \frac{1}{\Gamma(k)}\left[-2k\epsilon\Gamma(k) + (1+3\epsilon^2)\Gamma(k+2) \right.$$
$$\left. -4\epsilon(1+\epsilon^2)\Gamma(k+3) + 2\Gamma(k+4)(1+10\epsilon^2+5\epsilon^4)\right]\,, \tag{14}$$

substitute (14) and (7) in (13) we have (9) □

4. Maximum Likelihood Estimation for the ESΓ Distribution

For estimating the parameters of the ESΓ distribution, we derive the likelihood equations which lead to the maximum likelihood estimators assuming the location parameter $\theta = 0$; this means we standardize the distribution by assuming $\theta = 0$ and treat the other parameters $\beta, k,$ and ϵ as unknown.
Consider $X \sim \text{ES}\Gamma(0, \beta, k, \epsilon)$ be a random variable with a pdf given in (3), then the likelihood function is

$$L(\gamma) = \left(\frac{1}{2\Gamma(k)\beta^k}\right)^n \begin{cases} \prod_{i=1}^n \left(\frac{x_i^+}{(1-\epsilon)}\right)^{(k-1)} \; e^{-\frac{\sum_{i=1}^n x_i^+}{\beta(1-\epsilon)}} & \text{if } x \geq 0 \\ \prod_{i=1}^n \left(\frac{x_i^-}{(1+\epsilon)}\right)^{(k-1)} \; e^{-\frac{\sum_{i=1}^n x_i^-}{\beta(1+\epsilon)}} & \text{if } x < 0\,, \end{cases}$$

where $\gamma = (\beta, k, \epsilon)$,

$$x_i^+ = \begin{cases} x_i & \text{if } x_i \geq 0 \\ 0 & \text{o/w} \end{cases}\,, \tag{15}$$

$$x_i^- = \begin{cases} -x_i & \text{if } x_i \leq 0 \\ 0 & \text{o/w} \end{cases}\,. \tag{16}$$

and the $\log$ likelihood function is

$$\log L(\gamma) = -n\log(2) - n\log\Gamma(k) - nk\log(\beta) + (k-1)\sum_{i=1}^n \log(\frac{x_i^+}{(1-\epsilon)}) - \frac{\sum_{i=1}^n x_i^+}{\beta(1-\epsilon)}$$
$$+ (k-1)\sum_{i=1}^n \log(\frac{x_i^-}{(1+\epsilon)}) - \frac{\sum_{i=1}^n x_i^-}{\beta(1+\epsilon)}\,. \tag{17}$$

Maximizing (17) leads to the MLE of β

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i^+(1+\hat{\epsilon}) + \sum_{i=1}^n x_i^-(1-\hat{\epsilon})}{n\hat{k}(1-\hat{\epsilon}^2)}\,,$$

and the MLEs of k and ϵ are solved numerically.

5. Method of Moments Estimation (MME)

Since the ESΓ distribution consists of four parameters, we find the MME estimates by considering two cases:

1. Let θ and β be unknown and assume the parameters k and ϵ are known and let

$$m_1 = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \ , \tag{18}$$

using (6) and (7), we have the MMEs for θ and β are, respectively

$$\tilde{\theta} = \bar{x} + 2k\tilde{\beta}\epsilon \tag{19}$$

and

$$\tilde{\beta} = \frac{s}{\sqrt{k\left[(k+1)(1+3\epsilon^2) - 4k\epsilon^2\right]}} \ ,$$

substitute $\tilde{\beta}$ in (19), we have

$$\tilde{\theta} = \bar{x} + \frac{2\sqrt{k}\epsilon s}{\sqrt{(k+1)(1+3\epsilon^2) - 4k\epsilon^2}} \ ,$$

where $\bar{x}$ and s^2 are the sample mean and variance, respectively.

2. Let k and β be unknown and assume θ and ϵ are known, the moment estimators for β and k are, respectively

suppose $\theta = 0$, since it is known, we have

$$m_1 = (1/n)\sum_{i=1}^{n} x_i = -2\tilde{k}\tilde{\beta}\epsilon \Longrightarrow \bar{x} = -2\tilde{k}\tilde{\beta}\epsilon \Longrightarrow \tilde{k} = \frac{-\bar{x}}{2\tilde{\beta}\epsilon} \tag{20}$$

and the second moment with respect to sample is

$$\begin{aligned} m_2 = (1/n)\sum_{i=1}^{n} x_i^2 &= \tilde{\beta}^2\tilde{k}\left[(\tilde{k}+1)(1+3\epsilon^2)\right] \\ &= \tilde{\beta}^2\left(\frac{-\bar{x}}{2\tilde{\beta}\epsilon}\right)\left[\left(\frac{-\bar{x}}{2\tilde{\beta}\epsilon}+1\right)(1+3\epsilon^2)\right] \\ &= \left[\frac{\bar{x}^2 - 2\tilde{\beta}\epsilon\bar{x}}{4\epsilon^2}\right](1+3\epsilon^2) \end{aligned}$$

$$4\epsilon^2\sum_{i=1}^{n} x_i^2 = n(1+3\epsilon^2)\left[\bar{x}^2 - 2\tilde{\beta}\epsilon\bar{x}\right] \ ;$$

therefore,

$$\begin{aligned} \tilde{\beta} &= \frac{n\bar{x}^2(1+3\epsilon^2) - 4\epsilon^2\sum_{i=1}^{n} x_i^2}{2n\,\epsilon\bar{x}(1+3\epsilon^2)} \\ &= \frac{\bar{x}}{2\epsilon} - \frac{2\epsilon\sum_{i=1}^{n} x_i^2}{2n\,\epsilon\bar{x}(1+3\epsilon^2)} \end{aligned} \tag{21}$$

substitute (21) in (20) we obtain the MME of k.

6. Moment Generating and Characteristic Functions of the ESΓ Distribution

The mgf of a random variable X is defined by

$$\mu_x(t) = E(e^{tx})\,, \quad -h < t < h\,, h > 0\ . \tag{22}$$

proposition 4. *If $X \sim ES\Gamma(0, \beta, k, \epsilon)$, then the mgf of X has the form*

$$\mu_x(t) = \frac{(1+\epsilon)}{2(1+t\beta(1+\epsilon))^k} + \frac{(1-\epsilon)}{2(1-t\beta(1-\epsilon))^k}\ . \tag{23}$$

Proof. Using (22), we have

$$\begin{aligned}\mu_x(t) &= \int_{-\infty}^{\infty} e^{(tx)} f_{ES\Gamma}(x)\,dx \\ &= \int_{-\infty}^{0} \frac{e^{tx}}{2\Gamma(k)\beta^k}\left(\frac{-x}{(1+\epsilon)}\right)^{k-1} e^{-}(\frac{-x}{\beta(1+\epsilon)})\,dx \\ &+ \int_{0}^{\infty} \frac{e^{tx}}{2\Gamma(k)\beta^k}\left(\frac{x}{(1-\epsilon)}\right)^{k-1} e^{-}(\frac{x}{\beta(1-\epsilon)})\,dx\,.\end{aligned} \tag{24}$$

Let

$$z = \frac{-x(1+t\beta(1+\epsilon))}{\beta(1+\epsilon)}$$

for $x < 0$ and

$$z = \frac{x(1-t\beta(1-\epsilon))}{\beta(1-\epsilon)}$$

for $x \geq 0$, then (24) becomes

$$\begin{aligned}\mu_x(t) &= \frac{1}{2\Gamma(k)\beta^k}\int_{-\infty}^{0}\left(\frac{z\beta(1+\epsilon)}{1+t\beta(1+\epsilon)}\right)^{k-1}\frac{\beta(1+\epsilon)}{(1+t\beta(1+\epsilon))}\,e^{-z}\,dz \\ &+ \frac{1}{2\Gamma(k)\beta^k}\int_{0}^{\infty}\left(\frac{z\beta(1-\epsilon)}{1-t\beta(1-\epsilon)}\right)^{k-1}\frac{\beta(1-\epsilon)}{(1-t\beta(1-\epsilon))}\,e^{-z}\,dz \\ &= \frac{(1+\epsilon)}{2(1+t\beta(1+\epsilon))^k} + \frac{(1-\epsilon)}{2(1-t\beta(1-\epsilon))^k}\,.\end{aligned}$$

□

Note that when $\epsilon = 0$, (23) becomes the mgf of the reflected Γ distribution, while when $k = 1$ the mgf of ESL is retrieved. Also it can be shown when the mgf exists the rth derivative exists and the rth moment at $t = 0$ can be obtained.

proposition 5. *If $X \sim ES\Gamma\ (0, \beta, k, \epsilon)$, then the characteristic function of X is*

$$\phi_x(t) = \frac{(1+\epsilon)}{2(1+it\beta(1+\epsilon))^k} + \frac{(1-\epsilon)}{2(1-it\beta(1-\epsilon))^k}\ . \tag{25}$$

7. Fisher Information Matrix for the ESΓ Distribution

The Fisher information matrix plays a basic role in the asymptotic theory of the MLE's and in calculating the covariance matrices associated with it. The Fisher matrix could be computed from the expected values of the second partial derivatives of the $\log f(x;\gamma)$ as the form $I(\gamma) = -E[\frac{\partial^2 \log f(x;\gamma)}{\partial \gamma_i \gamma_j}]$

proposition 6. *If $X \sim ES\Gamma(0, \beta, k, \epsilon)$, then the Fisher information matrix for X, under regularity conditions, is*

$$I(\gamma) = -E[\frac{\partial^2 \log f(x;\gamma)}{\partial \gamma_i \gamma_j}] = \begin{bmatrix} \frac{k}{\beta^2} & \frac{1}{\beta} & 0 \\ \frac{1}{\beta} & \psi'(k) & \frac{\epsilon}{1-\epsilon^2} \\ 0 & \frac{\epsilon}{1-\epsilon^2} & R(\epsilon) \end{bmatrix},$$

where $\gamma = (\beta, k, \epsilon)$, for $i, j = 1, 2, 3$, $\psi'(k) = \frac{\Gamma''(k)}{\Gamma(k)}$ is the trigamma function, and $R(\epsilon) = \frac{2k\epsilon(1-\epsilon^2)-(k-1)(1+\epsilon^2)}{(1+\epsilon)^2(1-\epsilon)^2}$.

Proof. The proof is straightforward integration. □

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