Forecasting the U.S. Population with the Gompertz Growth Curve

Peter Pflaumer[1]
[1]Department of Statistics, Technical University of Dortmund, Vogelpothsweg 87, Dortmund 44227, Germany

Abstract
Population forecasts have recently received a great deal of attention. They are widely used for planning and policy purposes. In this paper, the Gompertz growth curve is proposed to forecast the U.S. population. In order to evaluate its forecast error, population estimates from 1890 to 2010 are compared with the corresponding predictions for a variety of launch years, estimation periods, and forecast horizons. Various descriptive measures of these forecast errors are presented and compared with the accuracy of forecasts made with the cohort component method (e.g., the U.S. Census Bureau) and other traditional time series models. These models include quadratic and cubic trends, which were used by statisticians at the end of the 19th century (Pritchett and Stevens). The measures of errors considered are based on the differences between the projected and the actual annual growth rate. It turns out that the forecast accuracies of the models differ greatly. The accuracy of some simple time series models is better than the accuracy of more complex models.

Key Words: Demography, Forecast Accuracy, Population Projection, Times Series

1. Introduction

The need for population forecasts is hardly disputed. In politics, in public administration, and in business, far-reaching decisions are made which depend on the future development of the population. The reliability of population predictions is influenced, however, by a multitude of factors. If long-term population forecasts are to serve as a rational basis for decision-making, then one needs to have an idea of their uncertainty. The smaller this uncertainty is, the more willing people will be to make decisions that are dependent on demographic factors (e.g., a decision to stabilize the financing of old age pensions).
In this paper, the Gompertz growth curve is proposed to forecast the U.S. population. In order to evaluate the forecast error, population estimates from 1890 to 2010 are compared with the corresponding predictions for a variety of launch years, base periods, and forecast horizons. Various descriptive measures of the forecast errors are presented and compared with the accuracy of forecasts made with the cohort component method (e.g., the U.S. Census Bureau) and other traditional time series models.

2. The Gompertz Growth Curve

A population can be forecast by the following general traditional time series model

$$P(T) = P(0) \cdot e^{\int_0^T r(t)dt} ,$$

where P(T) is the population at time T, P(0) is the base population at time 0, and r(t) is the continuous growth rate at time t.
If a declining exponential growth rate is assumed, viz.,
$r(t) = A \cdot e^{-k \cdot t}$ with A > 0, k > 0,
one obtains through integration

$$P(t) = P(0) \cdot e^{\frac{A}{k} - \frac{A}{k} \cdot \exp(-k \cdot t)} = C \cdot e^{-\frac{A}{k} \cdot \exp(-k \cdot t)}$$

with $C = P(0)e^{\frac{A}{k}}$. Here, C is the saturation level, since $\lim_{t \to \infty} C \cdot e^{-\frac{A}{k} \cdot \exp(-k \cdot t)} = C$.

The population at time 0 is given by $P(0) = Ce^{-\frac{A}{k}}$.
This function is called the Gompertz growth curve. It has been used by Winsor (1932) and other authors since 1926 (see Winsor, 1932, p. 7) as a growth curve, both for biological and economic phenomena[1]. It is a modification of the famous Gompertz law of 1825, which states that the force of mortality increases exponentially with age.
In terms of actuarial notation, this formula can be expressed as

$$\mu(x) = A \cdot e^{kx},$$

where $\mu(x)$ is the force of mortality, A > 0, k > 0, A is the general mortality level, x is the age, and k is the age-specific growth rate of the force of mortality. Since

$$\mu(x) = -\frac{d\ l(x)}{dx} \cdot \frac{1}{l(x)} = -\frac{d\ \ln l(x)}{dx},$$

one gets through integration the survivor function of the Gompertz distribution

$$l(x) = \exp\left(\frac{A}{k} - \frac{A}{k} e^{k \cdot x}\right) \quad x \geq 0.$$

Since with human populations $A << k$, the survivor function can be approximated for $-\infty < x < \infty$ by

$$l(x) = \exp\left(-\frac{A}{k} e^{k \cdot x}\right); \qquad F(x) = 1 - \exp\left(-\frac{A}{k} e^{k \cdot x}\right)$$ is called the Gumbel (minimum) distribution (cf., e.g., NIST/SEMATECH, 2011). This distribution is left-skewed.
In order to recognize the relationship of the Gompertz growth curve to the Gompertz distribution, the parameter C may be set to unity. In this case, one obtains an improper distribution function

$$F(t) = e^{-\frac{A}{k} \cdot \exp(-k \cdot t)} \text{ for } t \geq 0, \text{ since } F(0) = e^{-\frac{A}{k}}.$$

This improper distribution function can be modified in order to obtain the proper distribution function:

$$F(t) = \frac{e^{-\frac{A}{k} \cdot \exp(-k \cdot t)} - e^{-\frac{A}{k}}}{1 - e^{-\frac{A}{k}}} \text{ for } t \geq 0.$$

Its growth rate is

[1] I do not know of an application of the Gompertz growth curve in population forecasting with the exception of my own analysis (cf. Pflaumer, 1988).

$$r(t)=\frac{A\cdot e^{-k\cdot t}}{1-e^{\frac{A}{k}\cdot\exp(-k\cdot t)-\frac{A}{k}}}\quad \text{for } t\geq 0$$

with $\lim_{t\to 0}=\infty$ and $\lim_{t\to\infty}=0$. With increasing t, the growth rate approaches the exponential growth rate $r(t)=\frac{A}{1-e^{-\frac{A}{k}}}\cdot e^{-k\cdot t}>A\cdot e^{-k\cdot t}$.

A change of the domain leads also to a proper distribution function, which is the right-skewed Gumbel (maximum) distribution

$$F(t)=e^{-\frac{A}{k}\cdot\exp(-k\cdot t)}\quad -\infty<t<\infty .$$

The mathematical properties of the Gompertz growth curve were given in Winsor (1932). Differentiating the function twice, one gets the point of inflection

$m=\frac{\ln\left(\frac{A}{k}\right)}{k}$. The ordinate at the point of inflection is $\frac{C}{e}$, i.e., when approximately 37% of the final growth has been reached. At time t = m, the population increase is maximal. Near the point of inflection, the function can be approximated through a Taylor series expansion by

$$P(t)\approx\frac{C}{e}\cdot\left(k\cdot\left(t-m\right)+1\right).$$

A substitution yields an easy to interpret formula for the population trajectory,

$$P(t)=C\cdot e^{-\exp\left(-k\left(t-m\right)\right)},$$

where

C = upper asymptote (saturation level),

m = the time of maximum increase,

k = the rate of decrease of the growth rate.

The growth rate is $r(t)=A\cdot\exp(-k\cdot t)=k\cdot e^{-k\cdot(x-m)}=k\cdot e^{k\cdot m}e^{-k\cdot t}=r(0)\cdot e^{-k\cdot t}$.

Since $\ln P(t)-\ln C=-\frac{A}{k}\cdot e^{-k\cdot t}$, it is possible to describe the growth rate as a function of the population size P(t), that is $r(t)=k\cdot\ln\left(\frac{C}{P(t)}\right)=k\cdot\left(\ln C-\ln P(t)\right)$. The closer the population is at the saturation level, the lower the growth rate will be.

3. Empirical Analysis and Forecasts

3.1 The Gompertz Growth Curve

The analysis is carried out with decennial U.S. population data from 1790 to 2010 (cf. Table 2). The parameters have been estimated by a nonlinear method of least squares using the decennial population figures P(t) between the years 1790 and 2010 as a function of $t=\frac{year-1790}{10}$ or, $t=0,1,2,....21,22$.

Regression estimates were made based on n = 23 observations. The results are seen in Table 1. All estimators correspond to the theoretical assumptions, and are statistically significant. The model explains the development of the population between 1790 and 2010 on the whole very well, although the population totals in 2000 and 2010 have been underestimated by nearly 1% (cf. Table 2). The saturation level is about 1.37 billion; the population growth rate decreases on the average by roughly 6% every ten years (k=0.0613); the point of inflection is predicted for 2076 ($= 1790 + 28.6 \cdot 10$), which is the year with the maximum absolute population growth.

Table 1: Estimates of the Gompertz growth curve between 1790 (t = 0) and 2010 (t = 22)

Parameter	Estimate	t-Value
C	1370.0	6.73
k	0.0613	14.69
m	28.576	14.19
Cases incl.	23	

Nonlinear estimation

Table 2: Actual and estimated population

Year	**1790**	**1800**	**1810**	**1820**	**1830**	**1840**	**1850**	**1860**
Actual Pop.	3.93	5.31	7.24	9.64	12.87	17.07	23.19	31.44
Gompertz	4.30	6.10	8.40	11.40	15.10	19.80	25.40	32.20
Year	**1870**	**1880**	**1890**	**1900**	**1910**	**1920**	**1930**	**1940**
Actual Pop.	38.56	50.19	62.98	76.21	92.23	106.02	123.20	132.16
Gompertz	40.30	49.70	60.50	72.80	86.70	102.10	119.20	137.80
Year		**1950**	**1960**	**1970**	**1980**	**1990**	**2000**	**2010**
Actual Pop.		151.33	179.32	203.21	226.55	248.71	281.42	309.05
Gompertz		157.90	179.50	202.60	227.00	252.50	279.20	306.90

Model forecasts are compared with forecasts of the U.S. Census Bureau in Table 3. Their projections are based on Census 2000 and were produced using a cohort-component method. The components of change were projected into the future based on past trends. The projections cover the period 2000–2050. (U.S. Census Bureau, 2010). Between 2010 and 2050, the Census Bureau projects a growth of the U.S. population from 310 million to 439 million, an increase of 42%. The Gompertz growth curve projects, with a base population of only 307 million, a population of 425 million in 2050, an increase of 38%. The 99% prediction intervals of the Gompertz curve contain the low and the middle alternatives of the Census forecasts. In 2100, the population is projected to grow to 579 million, an increase of 89%. The fact that the population is also expected to become much older can only be concluded with the cohort-component method.

3.2 Alternative Simple Trend Models

The growth of the U.S. population has been forecast by simple trend models before. The first polynomial model was developed by the professor of mathematics and astronomy,

Table 3: Gompertz growth curve forecasts and U.S. Census Bureau forecasts

Year	Point Forecast	99% CI (LL)	99% CI (UL)	Census Forecast 2008	Census Forecast 2008 (low)	Census Forecast 2008 (high)
2010	306.90	295.60	318.21	310.23	308.28	312.50
2020	335.43	322.34	348.53	341.39	336.84	346.69
2030	364.69	348.91	380.46	373.50	365.68	382.61
2040	394.52	375.16	413.89	405.66	393.86	419.40
2050	424.81	400.98	448.64	439.01	422.55	458.18
2060	455.42	426.30	484.54			
2070	486.22	451.01	521.42			
2080	517.09	475.06	559.11			
2090	547.91	498.36	597.45			
2100	578.57	520.85	636.29			

Source: Own Calculations and U.S. Census Bureau (2009): National Projections; Low Net International Migration Series; High Net International Migration Series

Henry Smith Pritchett[2] (1891, 1900). By plotting and investigating the data from 1790 to 1880, he concluded that the growth of the U.S. population can be best explained by a cubic polynomial of time. By the use of his formula, he obtained decennial forecasts from 1910 to 2000. These forecasts were not bad for the first few decades. For longer periods, the model overestimated the actual population. He also made predictions for unusually long periods (2500: 11.9 billion; 3000: 40.9 billion). A similar approach is due to the professor of physics of the University of Maine, James S. Stevens (1900, 1910). He fitted a parabolic function of time to the population data between 1790 and 1880. His long-term forecasts are much better. For 2000, e.g., he predicted a population of 287.8 million, which is only slightly above the actual population. His forecasts for 2500 and 3000 are 3.4 and 10.1 billion people. Formally, the Stevens and the Pritchett model can be represented by

$$P(t) = a + b \cdot t + c \cdot t^2$$

and

$$P(t) = a + b \cdot t + c \cdot t^2 + d \cdot t^3,$$

with the growth rates

$$r(t) = \frac{\frac{dP(t)}{dt}}{P(t)} = \frac{b + 2ct}{a + bt + ct^2}$$

and

$$r(t) = \frac{\frac{dP(t)}{dt}}{P(t)} = \frac{b + 2ct + 3dt^2}{a + bt + ct^2 + dt^3}.$$

[2] Pritchett (1857–1939) was President of the Massachusetts Institute of Technology from 1900 to 1907 and wrote his doctoral thesis in Munich, Germany:
Über die Verfinsterungen der Saturntrabanten, München, Univ., Diss., 1895.

Although the polynomial functions do not have a saturation level, their growth rates approach zero with increasing time t.
The earliest population projections were made by assuming an exponential or geometric growth model with constant growth rate r: $P(T) = P(0) \cdot e^{r \cdot T}$. A good example of this kind of model is the forecast of the clergyman and Harvard professor E. Wigglesworth in 1775. In this study, "Calculations on American Populations," he made a long-term forecast for the population of the "British Colonies." He had observed that the population of his country doubled approximately every 25 years, which corresponded to a yearly growth rate of 2.8%. If one assumes, as Wigglesworth did, an initial population of 2.5 million and a continued constant growth rate of 2.8%, then one arrives at a forecast of 640 million people for the year 1975. With this figure, however, Wigglesworth considerably overestimated the actual population, which was less than 220 million in 1975. In the long term, the geometric trend model will considerably overestimate the population if the population growth rates are declining, which is the usual case.
The biologists R. Rearl and L.-J. Reed (1920) used the S-shaped logistic curve in order to produce long-term forecasts for the population of the United States. They predicted a saturation level of approximately 196 million. This population size was already surpassed in the mid 1960´s.
The logistic population growth curve goes back to Verhulst (1838). However, his work was ignored and eventually forgotten. Empirical investigations in general show that the population was significantly underestimated by the logistic function after as little as 30 years, e.g., Keyfitz (1979). In the long run, the application of the logistic model leads to an underestimation of the population size. Relevant formulas of the logistic curve are

$$P(t) = \frac{S}{1 + be^{-kt}}, \qquad S > 0, k > 0, b > 0, \; S = \text{saturation level},$$

$$\frac{dP(t)}{dt} = k \cdot P_t \cdot \left(1 - \frac{P(t)}{S}\right), \qquad r(t) = k \cdot \left(1 - \frac{P(t)}{S}\right) = k \cdot \frac{b}{e^{kt} + b}.$$

The logistic function is symmetric and has a point of inflection at S/2. The midpoint on the abscissa of the curve, where half of the saturation level has been achieved, is $t_{S/2} = \frac{\ln b}{k}$.

For the same estimation period with the same regressors t = 0, 1, … , 22 as for the Gompertz growth curve, the following results for the alternative models are presented in Table 4. All estimators are significant, and the coefficients of determinations of the polynomial models are near unity.
The forecasts by the alternative models are shown in Table 5. The differences between the models are remarkable. The exponential models project a very high population increase, whereas the logistic function projects a very low population increase in future years. The polynomial models yield similar projections as the Gompertz curve. This fact is not surprising, since the Gompertz curve behaves more or less like a polynomial curve up until the inflection point. For the present forecast period, the forecast of the Pritchett model is higher than the forecasts of the Stevens model and the Gompertz growth curve.

Table 4: Estimation results of the alternative models

Quadratic and Cubic Trend (Stevens and Pritchett)

Parameter	Estimate	t-Value	Parameter	Estimate	t-Value
a	74.1477	77.76	a	74.1477	86.08
b	13.607	142.21	b	13.1384	60.4
c	0.6783	41.94	c	0.6783	46.43
			d	0.00593	2.35
R-Squared	0.9991		R-Squared	0.9993	
Cases incl.	23		Cases incl.	23	

Logistic Function

Parameter	Estimate	t-Value
S	485.18	13.71
b	58.088	15.64
k	0.2081	23.23
Cases incl.	23	

Nonlinear estimation

Exponential Trend

Parameter	Estimate	t-Value
P(0)	16.327	10.45
k	0.1364	27.33
Cases incl.	23	

Nonlinear estimation

Table 5: Time series forecasts of the U.S. population (estimation period 1790–2010)

	Stevens			Pritchett			Logistic Function			Exponential Trend		
Year	Point	LL	UL	Point	LL	UL	Point	LL	UL	Point	LL	UL
2010	305.9	302.2	309.5	308.6	304.5	312.8	303.8	285.6	322.1	327.9	288.1	367.8
2020	335.1	324.6	345.6	339.7	328.7	350.8	326.8	305.2	348.3	375.8	331.8	419.9
2030	365.7	354.5	376.8	372.6	359.4	385.8	348.1	321.9	374.3	430.8	380.3	481.2
2040	397.6	385.7	409.5	407.3	391.3	423.3	367.6	335.8	399.5	493.7	434.1	553.2
2050	430.9	418.1	443.7	443.9	424.2	463.5	385.2	347.1	423.2	565.8	494.0	637.6
2060	465.5	451.6	479.4	482.3	458.3	506.3	400.7	356.2	445.2	648.5	560.7	736.3
2070	501.5	486.4	516.6	522.7	493.5	551.9	414.2	363.3	465.2	743.2	635.1	851.3
2080	538.8	522.5	555.2	565.0	529.8	600.2	425.9	368.8	483.1	851.8	718.4	985.2
2090	577.5	559.7	595.4	609.3	567.4	651.3	435.9	373.0	498.8	976.2	811.7	1140.8
2100	617.6	598.2	637.0	655.7	606.1	705.3	444.4	376.2	512.6	1118.9	916.3	1321.4

(LL and UL are lower and upper limits of 99% prediction intervals)

The point forecasts of the various models for the population in, e.g., 2050, range from 385 million to 566 million. Considering the time series of the growth rate of the U.S. population in the last two centuries in Figure 1, one can exclude the exponential model

and the logistic function as suitable forecasting models. Actual growth rates declined from 3% in the first decades of the estimation period to below 1 % in the last decades. The exponential model will probably overestimate the future growth rate, and the logistic function will underestimate it.
The three other models reflect the development of the growth rates quite well. Their forecasts range between 425 million (Gompertz growth curve) and 444 million (Pritchett model) in 2050. This interval contains the Census Bureau forecast of 439 million, too.
Which point forecast should be chosen, now? In order to answer this question, an ex post error analysis of the forecasts will be carried out in the next chapter. The exponential growth model will be dropped because of its unlikely forecasts.

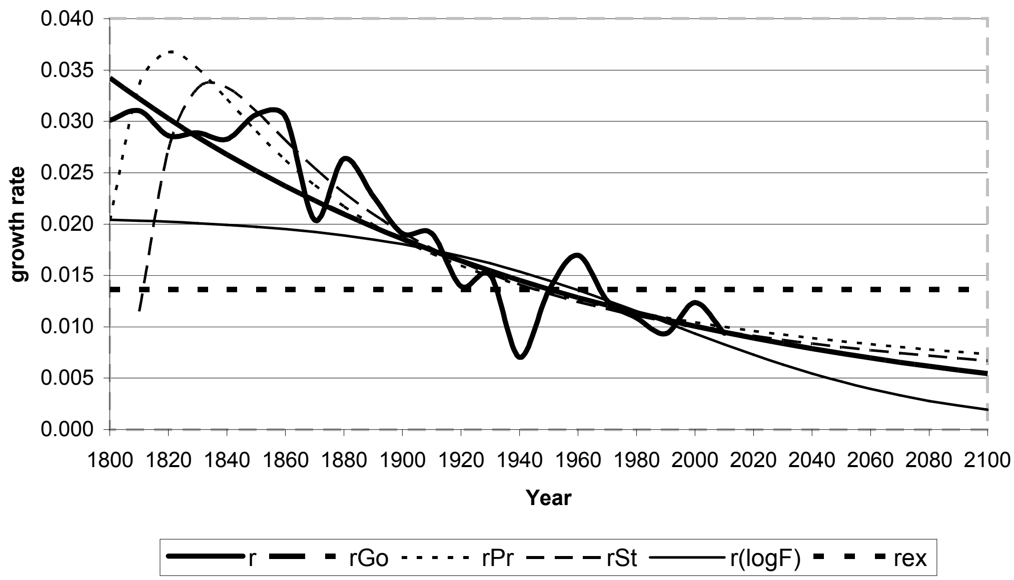

Figure 1: Yearly growth rates of the U.S. population

4. Evaluation of the Forecast Accuracy

4.1 Measures of the Forecast Error

There are several possibilities for studying the uncertainty of population forecasts: sensitivity analyses, forecast intervals in time series models, and stochastic component models (cf., e.g., Land, 1987). A further method of describing the uncertainty in population forecasts is based on the calculation of forecast error measures. They can aid in an ex post evaluation of a forecast. Under certain conditions, it is even possible to construct a forecast interval for future predictions on the basis of the distribution of past forecast errors (see, e.g., Keyfitz, 1981).
The measures of error considered here are based on the differences between the projected and the actual annual population growth rates. Keyfitz (1981) and Stoto (1983) have shown that this measure is independent of the population size and the length of the forecast period.
Let P_0 be the population at the beginning of the projection period, and P_T be the actual population T years later. It is easily shown that the actual average annual growth rate is

$$r = \frac{1}{T} \ln\left(P_T / P_0\right).$$

The projected average annual growth rate is

$$\hat{r} = \frac{1}{T} \ln\left(\hat{P}_T / \hat{P}_0\right),$$

where $\hat{P}_T$ is the projected population at time T and $\hat{P}_0$ is the estimated population at the beginning of the projection period. The following measures of error will be used.

(1) Logarithmic forecast error

$$d = \hat{r} - r;$$

if $\hat{P}_0 = P_0$, then

$$d = \frac{1}{T} \ln\left(\frac{\hat{P}_T}{P_T}\right).$$

(2) Average error or bias

$$\text{BIAS} = \frac{1}{n} \sum\left(\hat{r}_i - r_i\right) = \frac{1}{n} \sum d_i = \bar{d}.$$

(3) Mean square error

$$\text{MSE} = \frac{1}{n} \sum\left(\hat{r}_i - r_i\right)^2 = \frac{1}{n} \sum d_i^2$$

or

$$\text{MSE} = \text{BIAS}^2 + \sigma_d^2,$$

with

$$\sigma_d^2 = \frac{1}{n} \sum\left(d_i - \bar{d}\right)^2.$$

(4) Root mean square error

$$\text{RMSE} = \sqrt{\frac{1}{n} \sum\left(\hat{r}_i - r_i\right)^2} = \sqrt{\frac{1}{n} \sum d_i^2},$$

with

r_i = actual growth rate
$\hat{r}_i$ = projected growth rate
$d_i = \hat{r}_i - r_i$.

Empirical studies provide strong evidence that the RMSE is nearly invariant with respect to the time period over which the projection is made (cf. Keyfitz, 1981; Stoto, 1983; Pflaumer, 1993).

Keyfitz (1981) was the first to point out that the logarithmic forecast error is practically independent of the time period about which the prediction is made. In his study, he analyzed the errors of 1100 population forecasts. These forecasts had been made by the United Nations from 1958 to 1968 for all countries of the world having a population of more than one million. The most important result of Keyfitz's study is that the RMSE

varies by 0.3 percentage points for developed countries (low population growth) and 0.6 percentage points for developing countries (high population growth).
Jöckel & Pflaumer (1984), in connection with the study of Keyfitz, analyzed the errors of 360 population forecasts of the United Nations from the year 1958. They, too, arrived at the conclusion that the forecast error and the RMSE are relatively stable, but that the variance of the RMSE increases with the length of the forecast horizon. Indeed, this result also seems to be plausible. The increasing variance is an expression of the growing uncertainty that is caused by an extension of the forecast period.

4.2 Results of the Forecast Error Analysis

In order to get an idea of the accuracy of the Gompertz growth curve and the other simple time series models in the case of population forecasts, these models have been identified, estimated, and used for forecasting for various estimation periods, launch years, and forecast periods. Long-term ex post forecasts were made and compared with the actual population, and subsequently the logarithmic forecast error *d* and various error measures were calculated.
For each model, moving estimation intervals were selected, all beginning with the year 1790. The estimation results for the Gompertz model are shown in Table 6.

Table 6: Parameters of the Gompertz growth curve in different periods

Estimation Interval		Parameter			t-Value		
	n	C	k	m	C	k	m
1790-1890	11	11961.0	0.044	48.1	0.9	5.2	3.9
1790-1900	12	2812.1	0.057	33.6	1.7	7.4	5.7
1790-1910	13	1883.9	0.062	29.7	2.9	10.7	8.5
1790-1920	14	867.5	0.077	22.5	3.9	10.7	9.2
1790-1930	15	707.2	0.083	20.7	5.8	14.1	12.7
1790-1940	16	415.7	0.104	16.1	6.2	10.9	11.3
1790-1950	17	411.3	0.105	16.0	8.4	13.7	14.9
1790-1960	18	597.9	0.087	19.4	5.3	10.0	10.1
1790-1970	19	829.0	0.075	22.7	4.5	9.5	9.2
1790-1980	20	991.0	0.070	24.7	4.9	10.5	10.1
1790-1990	21	1014.0	0.069	24.9	6.1	12.6	12.4
1790-2000	22	1241.2	0.064	27.3	6.0	13.0	12.5
1790-2010	23	1370.0	0.061	28.6	6.7	14.7	14.2

On the basis of the estimated models, 10-year forecasts, 20-year forecasts, etc., were made and compared with the observed population numbers. Out of that, logarithmic forecast errors and other error measures were calculated. The forecast model using the estimation period from 1790 to 1890, e.g., allows of producing forecasts for 1900, 1910, 1920, …, 2000, and 2010, with forecast horizons of 10, 20, 30, ..., 100, and 110 years.
When interpreting the results, it is important to note that the measures are calculated out of 1 error with a forecast period of 120 years (estimation period 1790–1890), 2 errors with a forecast period of 110 years (estimation periods 1790-1890 and 1790-1900), and so on. Finally, it is possible to obtain 12 errors with a forecast period of 10 years (all estimation periods with exception of 1790–2010).
A benchmark for a forecast model is the so-called naïve model, which assumes no change in the future trend. Here, the naïve model is created by assuming a constant annual

growth rate for the total population, where this constant growth rate has been calculated from the last ten-year period of the launch year of the forecast. For example, if the launch year of the forecast is 1950, the annual population growth rate between 1940 and 1950 has been calculated and has been used for forecasts for future periods.
Table 7 shows the error measures of the various models. The bias of the logistic function forecasts is negative, which means that the population was significantly underestimated. The large RMSE of the logistic function, nearly 0.48 percentage points on average, is mainly explained by its large bias. The naïve model leads to a positive bias and an increasing RMSE; the accuracy of the forecasts decreases with increasing forecast length. In the case of short- and medium-term forecasts, the naïve model can nearly compete with the other models, but in the long run, the naïve model greatly overestimates the population, because of its constant growth rate. The Pritchett, Stevens, and Gompertz models are better than naïve forecasting. There is some evidence that the RMSE of the Pritchett model and the Gompertz growth curve are nearly invariant with respect to the length of the forecast period, whereas their bias is increasing, when the forecast period is more than 60 years. The simpler Stevens model yields surprisingly very good results. The bias fluctuates around zero, and its RMSE is always lower than the RMSE of the other models. The forecast accuracy improves relatively with the forecast period, since its RMSE decreases. The Gompertz growth curve and the Pritchett model have similar patterns in their error measures, albeit the accuracy of the Pritchett model is slightly better than that of the Gompertz growth curve. In summary, it should be noted that the following ranking applies with respect to the forecast accuracy: first, the Stevens model; second, the Pritchett model; and third, the Gompertz growth curve. The averages of their RMSEs over all forecast periods are: 0.09, 0.28, and 0.34 percentage points.

Table 7: Error measures for all models in dependence on the forecast horizon (percentage points)

	BIAS					RMSE				
Horizon	Gompertz	Pritchett	Stevens	Log. F	Naïve F.	Gompertz	Pritchett	Stevens	Log. F.	Naïve F.
10	-0.003	0.055	-0.043	-0.340	0.111	0.394	0.359	0.303	0.530	0.410
20	0.007	0.058	-0.020	-0.329	0.138	0.329	0.293	0.197	0.463	0.401
30	0.023	0.072	-0.003	-0.348	0.211	0.325	0.279	0.143	0.464	0.438
40	0.028	0.077	0.001	-0.389	0.263	0.328	0.272	0.111	0.486	0.465
50	0.040	0.088	0.007	-0.425	0.305	0.344	0.279	0.085	0.514	0.504
60	0.056	0.099	0.001	-0.458	0.278	0.331	0.268	0.064	0.527	0.493
70	0.107	0.145	0.001	-0.459	0.315	0.289	0.236	0.037	0.500	0.523
80	0.183	0.210	0.008	-0.441	0.509	0.271	0.234	0.025	0.462	0.556
90	0.239	0.243	0.005	-0.443	0.578	0.294	0.255	0.016	0.457	0.624
100	0.310	0.278	-0.001	-0.432	0.732	0.334	0.281	0.013	0.439	0.742
110	0.354	0.282	-0.020	-0.447	0.772	0.368	0.283	0.023	0.452	0.784
120	0.454	0.310	-0.029	-0.412	0.945	0.454	0.310	0.029	0.412	0.945

Table 8: Error measures for all models in dependence on the launch year (percentage points)

	BIAS					RMSE				
Launch Year	Gompertz	Pritchett	Stevens	Log. F	Naïve F.	Gompertz	Pritchett	Stevens	Log. F.	Naïve F.
1890	0.448	0.315	-0.046	-0.131	0.719	0.454	0.322	0.101	0.223	0.745
1900	0.308	0.278	-0.002	-0.262	0.453	0.318	0.289	0.102	0.315	0.489
1910	0.318	0.334	0.086	-0.207	0.618	0.337	0.346	0.118	0.304	0.623
1920	0.135	0.218	0.096	-0.365	0.138	0.200	0.250	0.154	0.422	0.175
1930	0.115	0.218	0.158	-0.343	0.383	0.292	0.331	0.287	0.458	0.419
1940	-0.284	-0.180	-0.014	-0.716	-0.640	0.311	0.216	0.144	0.725	0.648
1950	-0.483	-0.379	-0.175	-0.882	-0.011	0.494	0.395	0.190	0.891	0.175
1960	-0.327	-0.210	-0.153	-0.669	0.553	0.339	0.229	0.173	0.677	0.556
1970	-0.183	-0.073	-0.108	-0.467	0.193	0.189	0.087	0.120	0.470	0.195
1980	-0.111	-0.004	-0.076	-0.370	0.069	0.125	0.061	0.090	0.373	0.094
1990	-0.247	-0.129	-0.190	-0.512	-0.228	0.255	0.140	0.201	0.517	0.240
2000	-0.150	-0.027	-0.153	-0.411	0.299	0.150	0.027	0.153	0.411	0.299

Table 8 shows error measures in dependence on the various launch years of the forecasts. The error measures are calculated from 12 errors of forecast periods from 10 to 120 years (launch year 1890), 11 errors of forecast periods from 10 to 110 years (launch year 1900), and so on. Finally, 2 forecast errors of forecast periods from 20 to 10 years (launch year 1990), and 1 error of the forecast period of 10 years (launch year 2000) were used. When the error measures are compared in Tables 7 and 8, it becomes apparent that the forecast error primarily depends on the time at which the forecast was made and not so much on the length of the forecast period. Especially large are the errors with launch years 1930 to 1950. The models greatly underestimated the future population development. This underestimation with large negative biases logically leads to high values of the RMSE. Also, the demographers of the 1930´s and 1940´s underestimated future population totals despite their comparatively sophisticated methods. The reason for these incorrect predictions lay in the assumption that the low fertility level of these years would continue in the future. The baby boom of the 1950's and the early 1960's was not predicted and therefore not included in any calculations either. Thus, for example, the respected demographers Thompson and Whelpton (1943) underestimated the yearly population growth on average by one percentage point. The Stevens model outperformed the other models for nearly all launch years. For most launch years, the Pritchett model outperformed the Gompertz growth model, again, despite the similar error patterns. The averages of the RMSE were: 0.16 (Stevens model), 0.22 (Pritchett model), and 0.29 percentage points (Gompertz growth curve). The average RMSE of the naïve method was, at 0.39 percentage points, high, although it produced good results in some launch years.

At the end of this chapter, the performance of the time series models will be compared with that of the Census Bureau. Mulder (2002) published a comprehensive analysis of the accuracy of the Census Bureau. Her paper evaluates the accuracy of Census Bureau population forecasts and their components made between 1947 and 1994 for forecast periods of five, ten, fifteen and twenty years. Her naïve model assumes a constant growth rate, that of the launch year of the forecast (Mulder, 2002, p. 14). The results regarding the RMSE for multiple series are summarized in Table 9. The naïve model outperformed the forecasts of the Census Bureau in short-term forecasts, whereas in the other cases similar measures of the RMSE have been achieved.

Table 9: RMSE of U.S. Census Bureau population forecasts between 1947 and 1994 (percentage points)

Horizon	U.S. Census Bureau Forecast	Naïve
5	0.30	0.18
10	0.37	0.30
15	0.39	0.38
20	0.43	0.46

Source: Mulder (2002), Table 2 (multiple series)

In order to compare Mulder´s results with the previous results, error measures of the time series models were calculated for forecasts made between 1950 and 2000 (cf. Table 10)

Table 10: Error measures for all models in dependence on the forecast horizon from launch years 1950 to 2000 (percentage points)

	RMSE				
Horizon	Gompertz	Pritchett	Stevens	Log. F.	Naïve F.
10	0.392	0.307	0.230	0.703	0.303
20	0.303	0.221	0.148	0.597	0.274
30	0.279	0.191	0.113	0.590	0.316
40	0.297	0.194	0.104	0.637	0.356
50	0.359	0.247	0.119	0.728	0.438
60	0.413	0.298	0.115	0.805	0.164

Because of the differences in the launch years and the lengths of the forecast periods, the comparison is limited. But it can be assumed that the time series models yield forecast results which are at least not worse than those of the more complex model of the Census Bureau. This paradox, that simple models often outperform more complex models, has been extensively discussed in the demographic literature (cf., e.g., Keyfitz, 1981; Stoto & Schrier, 1982; Stoto, 1983; Pflaumer, 1992; Rogers, 1995; McNown et al., 1995; Long, 1995; Ahlburg, 1995).
Smith & Sincich (1992) mention as a reason for this paradox the difficulty of anticipating future trends in the components of the more complex models. They believe that the cohort-component projections are no more accurate than the trend and ratio projections because forecasting fertility, mortality, and migration is just as difficult as forecasting population changes.

5. Summary and Conclusion

In the present paper, the Gompertz growth curve was considered as a population forecasting model. It is a simple alternative to the more complex cohort-component method. Evaluating past forecast errors shows that its forecast accuracy is not worse than that of the Census Bureau, which uses the cohort-component-method. The Gompertz growth curve has a saturation level like the logistic function. The logistic function underestimates the population in the long run. This tendency can not be confirmed for the Gompertz growth curve at all. Simpler alternative time series models such as the Pritchett model and the Stevens models were considered, too. It turned out that these two models outperformed the Gompertz growth curve in the calculation of past forecast errors. The Stevens model, a parabolic trend model, achieved the best results. Pflaumer (1992) discussed the Box–Jenkins approach for forecasting the U.S. population. He concluded that the U.S. population can be satisfactorily described by an ARIMA(2,2,0) process and showed that this model is equivalent to a parabolic trend model or Stevens model when making long-term population forecasts. The superiority of the polynomial models over the Gompertz growth curve should not be generalized. Investigations with other population forecasts are still necessary.

It is remarkable that such a simple model as the Stevens model made such reliable forecasts during the last century. Should it be applied to forecast the future population of the United States? Obviously it is not suitable when an age-structured forecast is required. It is also not possible to decide a priori which forecasting model will perform best in the future. But the forecast accuracy of the past is one indicator for choosing a specific model. In addition to the forecast comparison, the time series models serve to investigate the plausibility of the assumptions of more complex models. Time series models should be regarded as a supplement to the more complex models. In the past, one could often observe a discrepancy between the forecasts of the Census Bureau, with its more complex model, and the simpler time series forecasts. Since right now both the Census Bureau and the time series models forecast show similar results for the future population of the U.S., it is to be hoped and to be expected that these forecasts will come true. In the year 2050, e.g., the middle or point forecasts are (in millions): Census Bureau, 439; Gompertz growth curve, 425; Pritchett model, 444; Stevens model, 431. The forecast range of the different methods is very small. Thus, from today´s perspective, it is to be expected that the Census Bureau's forecast is accurate.

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