Diagnosing and Correcting Balance Assessments in High Dimensions

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Joint work with Ben Hansen

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In either case, **assessing the similarity of treatment and control groups** ("balance") is an important component of study design.

Notation

- There are n **units** (people, clinics, schools)
- The p **variables** are collected in $\mathbf{x}$
- Each unit is assigned to **treatment** $Z_i = 1$ or **control** $Z_i = 0$.
- It will be useful to define

$$J_i = \frac{Z_i - \mathrm{P}(Z_i = 1)}{\mathrm{P}(Z_i = 1)\,\mathrm{P}(Z_i = 0)}$$

Mahalanobis Distance

Hansen and Bowers (2008) and Morgan and Rubin (2012) assess balance with a **Mahalanobis distance**, a normalized difference of group means $\mathbf{D}$:

$$M = \mathbf{D}'\text{Cov}\left(\mathbf{D}\right)^{-}\mathbf{D} = \mathbf{J}'\mathbf{x}\,\text{Cov}\left(\mathbf{x}'\mathbf{J}\right)^{-}\mathbf{x}'\mathbf{J}$$

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A common special case is **complete random assignment** with n_1 assigned to treatment and $n_0 = n - n_1$ to control:

$$M = \frac{n_1 n_0}{n}(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_0)' \, S^2(\mathbf{x})^{-1}(\bar{\mathbf{X}}_1 - \bar{\mathbf{X}}_0)$$

where $\mathbf{X}_1$ and $\mathbf{X}_0$ are group means and $S^2(\mathbf{x})$ is the sample covariance matrix.

Interpretation via PCA

Since we are **scaling by the inverse covariance matrix** there is a convenient representation of the Mahalanobis distance using **the principal components of $\mathbf{x}$**:

$$M = \mathbf{J}'\mathbf{x}(\mathbf{x}'\,\mathrm{E}\,(\mathbf{J}\mathbf{J}')\,\mathbf{x})^{-}\mathbf{x}'\mathbf{J} = \mathbf{J}'\mathbf{u}\mathbf{u}'\,\mathrm{E}\,(\mathbf{J}\mathbf{J}')^{-}\,\mathbf{u}\mathbf{u}'\mathbf{J},$$

where $\mathbf{u}$ is from the SVD of $\mathbf{x} = \mathbf{u}\mathbf{d}\mathbf{v}'$.

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As an added bonus, we now have a natural way of **ordering the covariates**.

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With the number of covariates p **being fixed** and n **tending to infinity**,

$$M \xrightarrow{\text{D}} \chi^2(p)$$

(Hansen and Bowers, 2008; Li et al., 2018)

Example: Homicide rates and public infrastructure

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- Intervention: Public works **transportation program**.
- Units: **48 neighborhoods** (25 with stations, 23 without) matched in **pairs and two triples**.
- Covariates: Survey responses, governmental data, mix of types (**48 with interactions**)
- Balance Assessment: Validating matching strategy created **comparable pairs and triples**.

Cerdá et al. (2012) Empirical Mahalanobis Distance

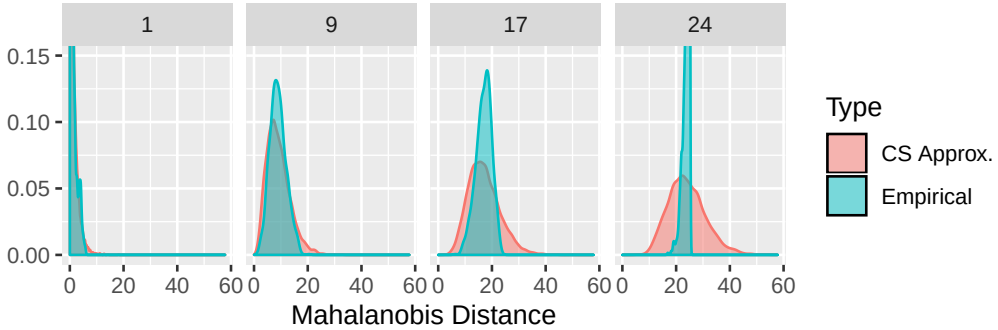

Figure 1: Mahalanobis distance distribution for 48 matched neighborhoods in reported in Cerdá et al. (2012)

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As the previous slides demonstrated, **selecting** $k < n$ **principal components** provides a **non-degenerate distribution**.

We define **reduced rank Mahalanobis distance** as:

$$M_k = \mathbf{J}'\mathbf{u}_k\mathbf{u}_k'\,\mathrm{E}\left(\mathbf{J}\mathbf{J}'\right)^{-}\mathbf{u}_k\mathbf{u}_k'\mathbf{J}$$

where $\mathbf{u}_k$ collects the first k columns of $\mathbf{u}$.

Picking k

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A natural analog in this case is **maximizing the (null) variance of M_k**.

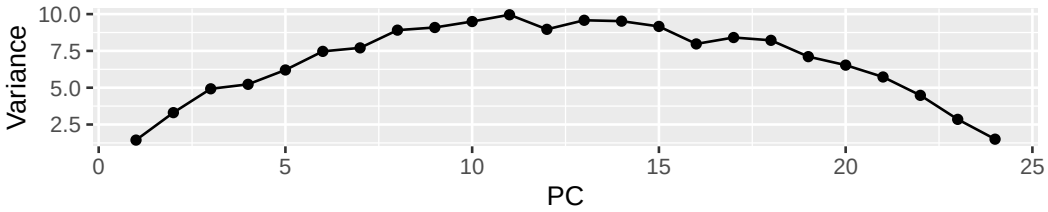

Figure 2: Variance of M_k for matched neighborhoods in Medellín

Satterthwaite Approximation

Rather than approximate the distribution of M_k as $\chi^2(k)$, we could use a correction that matches both **mean and variance** to a **scaled** χ^2:

$$M_k \approx a\chi^2(v) \Rightarrow v = \frac{2k^2}{\text{Var}\,(M_k)},\ a = \frac{k}{v}$$

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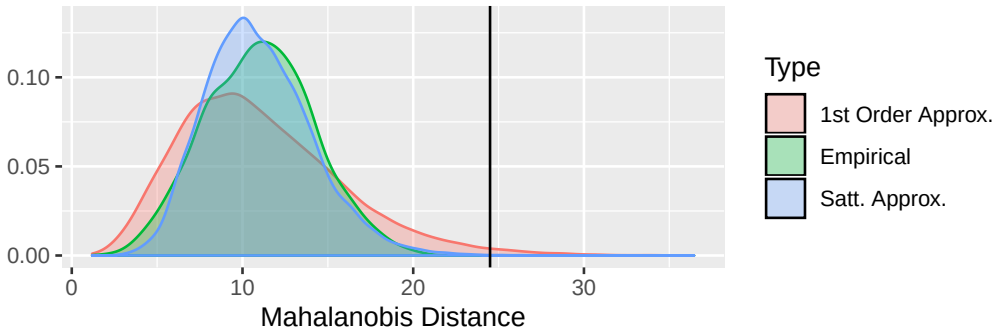
- Provides approximate cutoff for selecting "rerandomization" RCT designs
- Alternatively, use Monte Carlo sampling from $\mathbf{Z}$

Corrected Medellín Distribution



$m_{11} = 24.5$; $\mathrm{P}\left(M_{11} \geq m_{11}\right)$: 0.01088 (1st order approx.), 0.00048 (2nd order approx.), 0.001 (empirical, 5k samples)

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- **Reduced rank Mahalanobis distance** guarantees non-degenerate distribution.
- **Satterthwaite χ^2 approximation** superior and relatively simple.
- **In-progress**: extend `RItools` package for R, formalize justification for variance maximization, implications for design (both observational and RCTs)

Thank You!