Modeling Vector Random Fields on a Sphere: Application on Geomagnetic Field

Amy Kim

Debashis Paul, Thomas Lee, Tomoko Matsuo
University of California, Davis

atykim@ucdavis.edu

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Overview

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Motivation

- Prediction and modeling fluctuations in multivariate processes
- Interested in vector random fields on a sphere
- Unified framework for modeling both mean and residual fields
- Producing a meaningful decomposition of vector fields on a sphere

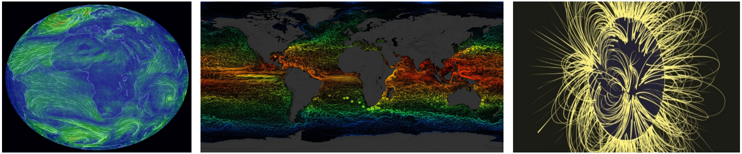

Figure: Wind Map [1], Ocean Current [2], Geomagnetic Field [3]

[1] https://www.theregister.co.uk/
[2] NASA
[3] https://www.sciencealert.com/

Helmholtz-Hodge Decomposition

Any vector field $\mathbf{v}$ can be decomposed as

$$\mathbf{v} = \mathbf{v}_{\text{curl-free}} + \mathbf{v}_{\text{div-free}} + \mathbf{v}_{\text{harmonic}} \tag{1}$$

where $\mathbf{v}_{\text{curl-free}}$ is a curl free, $\mathbf{v}_{\text{div-free}}$ is a divergence free, and $\mathbf{v}_{\text{harmonic}}$ is both divergence-free and curl-free and is the gradient of a harmonic functions.

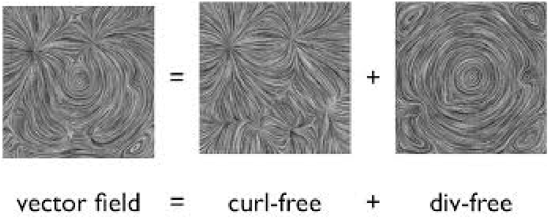

Figure: Decomposition of Vector Field [4]

[4] https://images.app.goo.gl/Yv79h7N2SeeYJVi96

Models for Multivariate Random Fields

Vector Random fields can be treated as multivariate processes.

- Multivariate Matérn Model (Gneiting et al. (2010), Apanasovich et al. (2012))
- Multivariate Random Fields on globe (Jun (2014), Jun (2011))
- Tangent Matérn Model (Fan et al. (2018))

Our Approach

Modeling vector random field on a sphere:

- Vector Spherical Harmonics: basis functions for vector fields defined on sphere, $\mathbb{S}^2$, where $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$
- Model mean and error fields simultaneously (Linear mixed effect model approach)
- Produce physically meaningful vector fields

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Spherical Coordinate System

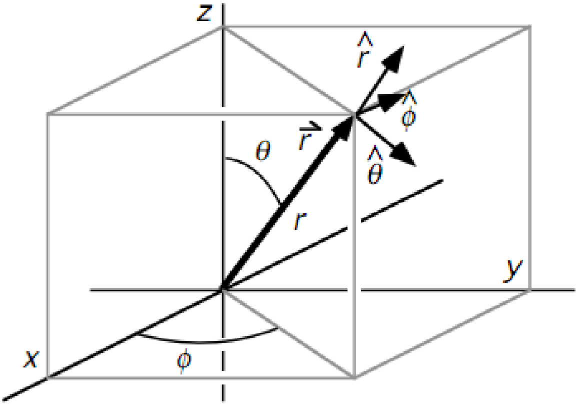
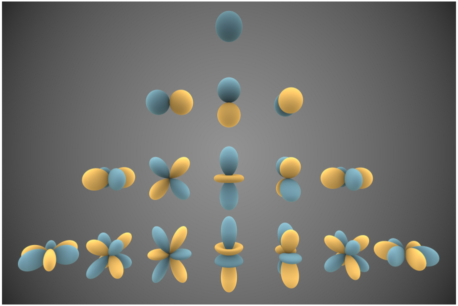


Figure: Spherical Coordinate System

Scalar Spherical Harmonics

For every integer $l = 0, 1, 2, \ldots,$ and every $m = -l, \ldots, l$, the complex **Scalar spherical harmonic function** $Y_{l,m}(\theta, \phi)$



Visual Presentation of SSH[5]

- Eigenfunctions of the Laplacian on $\mathbb{S}^2$
- Orthonormal basis of $L^2(\mathbb{S}^2)$

[5] By Inigo.quilez - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=32782753

Vector Spherical Harmonics

Given a scalar spherical harmonics $Y_{l,m}(\theta, \phi)$, define complex **Vector Spherical Harmonics** $\{\mathbf{Y}_{l,m}, \mathbf{B}_{l,m}, \mathbf{C}_{l,m}\}$:

- $\mathbf{Y}_{l,m} = Y_{l,m}\hat{\mathbf{r}}$
- $\mathbf{B}_{l,m} = \frac{1}{\sqrt{l(l+1)}}\nabla Y_{l,m}$
- $\mathbf{C}_{l,m} = -\hat{\mathbf{r}} \times \frac{1}{\sqrt{l(l+1)}}\nabla Y_{l,m} = -\hat{\mathbf{r}} \times \mathbf{B}_{l,m}$

$\text{Curl}\{\mathbf{B}_{l,m}\} = \nabla \times \mathbf{B}_{l,m} = 0, \quad \text{Div}\{\mathbf{C}_{l,m}\} = \nabla \cdot \mathbf{C}_{l,m} = 0$

VSH: Tangential Vector Random field modeling

Decomposition tangential vector fields on $\mathbb{S}^2$:

$$\mathbf{v} = \sum_{1 \leq l, |m| \leq l} \left[f^B_{l,m} \mathbf{B}_{l,m} + f^C_{l,m} \mathbf{C}_{l,m} \right] \tag{2}$$

$$\text{Curl}\{\mathbf{B}_{l,m}\} = \nabla \times \mathbf{B}_{l,m} = 0, \quad \text{Div}\{\mathbf{C}_{l,m}\} = \nabla \cdot \mathbf{C}_{l,m} = 0$$

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$$\begin{aligned}\mathbf{v} &= \sum_{1 \le l, |m| \le l} \left[f^B_{l,m} \mathbf{B}_{l,m} + f^C_{l,m} \mathbf{C}_{l,m} \right] \\ &= \underbrace{\sum_{1 \le l, |m| \le l} f^B_{l,m} \mathbf{B}_{l,m}}_{\text{Curl-Free}} + \underbrace{\sum_{1 \le l, |m| \le l} f^C_{l,m} \mathbf{C}_{l,m}}_{\text{Divergence-Free}}\end{aligned} \tag{2}$$

$$\text{Curl}\{\mathbf{B}_{l,m}\} = \nabla \times \mathbf{B}_{l,m} = 0, \quad \text{Div}\{\mathbf{C}_{l,m}\} = \nabla \cdot \mathbf{C}_{l,m} = 0$$

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SSH: Gaussian Random Fields

The Spectral representation of isotropic fields on $\mathbb{S}^2$(Marinucci & Peccati (2011))

A Gaussian isotropic random field $h(\theta, \phi)$ on $(\theta, \phi) \in \mathbb{S}^2$, with $\sum_l \sum_m E\|h^2(\theta, \phi)\| < \infty$, can be defined as

$$h(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(\theta, \phi)$$

where $f_{l,m}$ are Gaussian random variables, independent across l.

Tangential Vector Random Field Model on a Sphere

A Gaussian tangential vector random field on a sphere, $\{y(s) : s \in \mathbb{S}^2\}$, can be constructed of

$$y(s) = \sum_{l=1}^{L} \sum_{m=-l}^{l} \left[f_{l,m}^{B} \mathbf{B}_{l,m}(s) + f_{l,m}^{C} \mathbf{C}_{l,m}(s) \right] + \epsilon_s \quad (3)$$

where

- L: the maximum order of vector spherical harmonics
- $f_{l,m}^{B}, f_{l,m}^{C}$: random complex coefficients (conjugacy relationship $m, -m$)
- $y(s) \in \mathbb{R}$

Assumptions of TVRF on a Sphere

- $f^B_{l,m} \sim CN(0, \sigma^2_{B,l}),\ f^C_{l,m} \sim CN(0, \sigma^2_{C,l})$
  - Independence between $\{f^B_{l,m}\}, \{f^C_{l,m}\}$
  - The coefficients are independent across l
  - For the fixed l, the coefficients are independent identically distributed across m subject to conjugacy
  - $\sigma^2_{B,l} = l^{-\alpha_B}\sigma^2_B,\ \sigma^2_{C,l} = l^{-\alpha_C}\sigma^2_C$, and $\alpha_B, \alpha_C > 0$
- $\epsilon_s \sim N(0, \tau^2)$, independent across observed locations (s)
- $y(s)$ has Gaussian distribution.

Matrix Form

The observed TVRF $\mathbf{y} = [y^{\theta}(s_i)_{1 \le i \le n},\ y^{\phi}(s_i)_{1 \le i \le n}]^T$ is

$$\mathbf{y} = \mathbf{Z}\mathbf{f} + \epsilon \tag{4}$$

where $y(s_i) = y^{\theta}(s_i)\hat{\boldsymbol{\theta}} + y^{\phi}(s_i)\hat{\phi}$, and

- $\mathbf{y}, \epsilon$: length $2n$ vectors
- $\mathbf{Z}$: $2n \times p$ matrix of VSH
- $\mathbf{f}$ is a $p \times 1$ vector of random VSH coefficients
- $p = 2(L^2 + 2L) + 2(L^2 + L)$

Then $\mathbf{y} \sim N(\mathbf{0}, V)$ where $V = \mathbf{Z}G\mathbf{Z}^T + \tau^2 I_{2n}$

VSH: Simulated Field

Let **v** be a tangential vector field on a bounded domain in $\mathbb{S}^2$, then it can be expressed as a sum of a curl-free and a divergence-free vector fields:

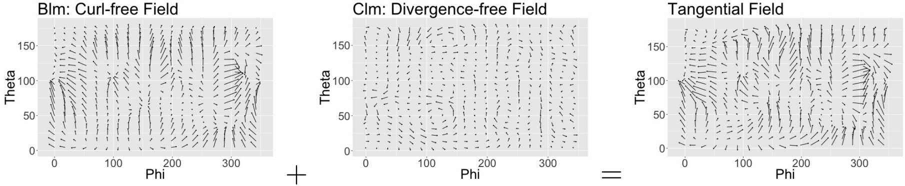

Figure: Simulated Tangential Vector Field on $\mathbb{S}^2$: $\sigma_B^2 = 20, \sigma_C^2 = 10$

Simulations

- Estimation results 500 data sets on 20×20 equiangular grid

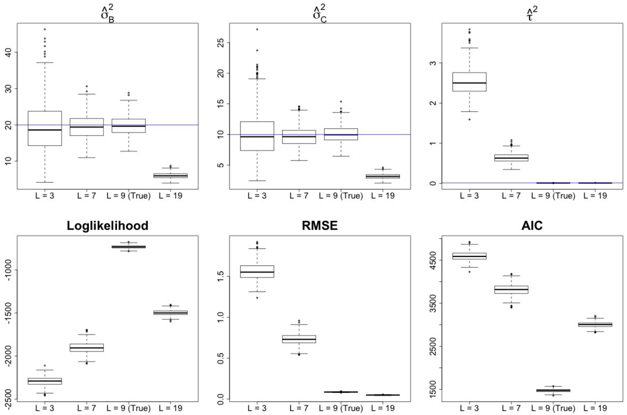

Figure: $L_{\text{True}} = 9, \sigma_B^2 = 20, \sigma_C^2 = 10, \tau^2 = 0.01$ with known $\alpha_B = \alpha_C = 2$.

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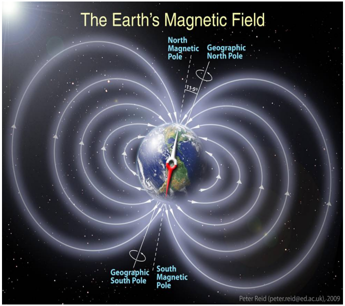
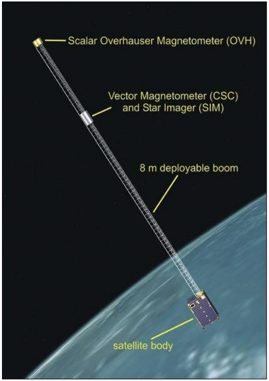
Geomagnetic Field



- Earth's magnetic field: **B**
- $\mathbf{B} = -\nabla \mathbf{V}$ (Ampere's Law, Maxwell's equation)
- Curl-free field
- Satellites Data: Magsat, Ørsted, CHAMP, Swarm

Ørsted Satellites



- Data collected from Feb 1999
- High precision mapping the Earth's magnetic fields
- Attitude error - anisotropic

Proposed Model: Geomagnetic Field

Let $\mathbf{Y}(s,t)$ be the tangential geomagnetic field at location s and time t. Then,

$$\mathbf{Y}(s,t) = \boldsymbol{\mu}(s,t) + \varphi(\theta,t)\mathbf{y}(s) \tag{5}$$

where $\boldsymbol{\mu}(s,t)$ is non-random time varying trend, $\varphi(\theta,t)\mathbf{y}(s,t)$ is random stochastic fluctuation.

- $\boldsymbol{\mu}(s,t) = \sum_{l=1}^{L_\mu} \sum_m \{g_{l,m}^B(t)\mathbf{B}_{l,m}(s) + g_{l,m}^C(t)\mathbf{C}_{l,m}(s)\}$
    - $\{g_{l,m}^B(t)\}, \{g_{l,m}^C(t)\}$ are represented in spline basis
- $\varphi(\theta,t)$: axially symmetric scaling function
- $\mathbf{y}(s,t) = \sum_{l=1}^{L_R} \sum_m \left[f_{l,m}^B(t)\mathbf{B}_{l,m}(s) + f_{l,m}^C(t)\mathbf{C}_{l,m}(s)\right] + \epsilon(s,t)$
    - $\{f_{l,m}^B(t)\}, \{f_{l,m}^C(t)\}$ are independent across t

Proposed Model: Geomagnetic Field

- $\varphi^x(\theta, t) = \sqrt{\int \|\boldsymbol{\mu}^x(\theta, \phi, t)\|^2 d\phi}$ where $x = \theta, \phi$.

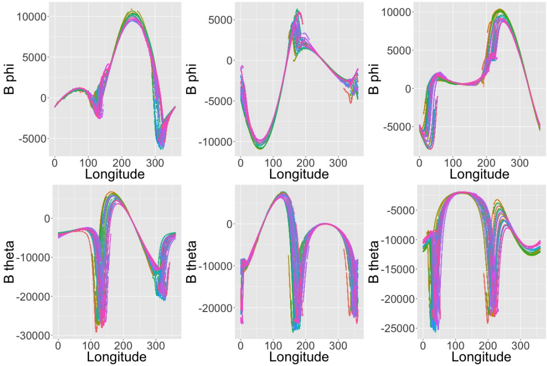


Figure: B^{ϕ}(first row) vs. ϕ and B^{θ}(second row) vs. ϕ on three different trajectory groups(each columns).

Results and Comparison to other methods

Comparing our fitted model (VRF):

- $\sum_{t=1}^{15} n_t = 24601$
- $J = 5, L_R = 19$
- Optimization: $\hat{\sigma}_B^2 = 0.4846, \hat{\sigma}_C^2 = 0.02076, \hat{\tau}^2 = 0.002429$
- AIC: $\hat{L}_\mu = 3, \hat{\alpha}_B = 3, \hat{\alpha}_C = 2$

With

- Random Forest (RF)
- Generalized Boosted Regression model (GBM)
- Linear Regression with Vector Spherical Harmonics $L_\mu = 3$ with considering time (VSH)

Results and Comparison to other methods

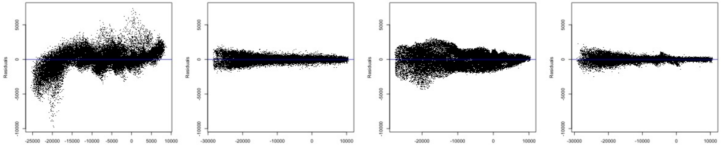


Figure: Fitted vs. Residual Plots

	RF	GBM	VSH	VRF
rmse	1200.1718	351.1564	920.4255	286.5607
prmse	1189.4378	392.3918	920.0997	379.1270

Table: Comparison of squared-root mean squared error (rmse: `sqrt(mean(residuals`^{2}`))`) and predicted squared-root squared error(prmse: dataset not used for estimating parameters.) of each four models

Vector Field Plot

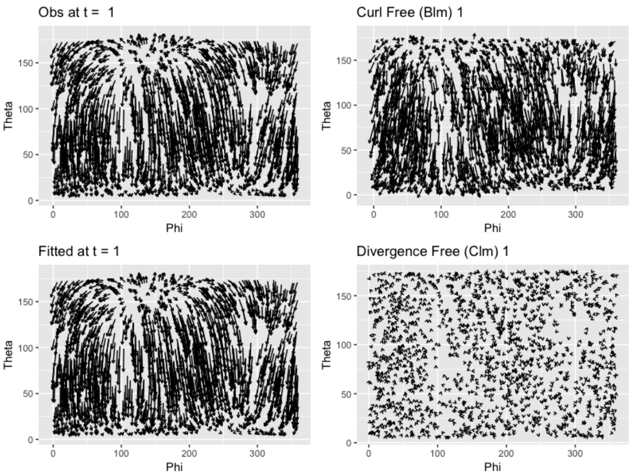


Figure: Geomagnetic field plot at the first time period

Vector Field Plot

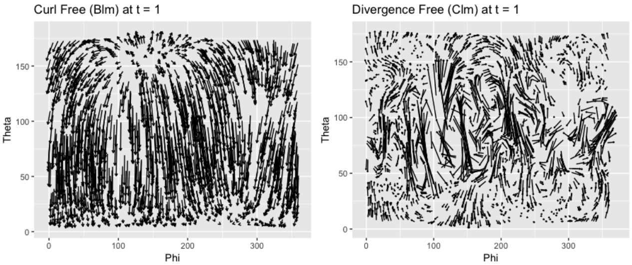


Figure: Different scale: Clm field is enlarged 3 times scale

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Final Remarks

- Parametric model for tangential vector random fields on the surface of sphere through a vector spherical harmonics representation.
- The vector random field models(random effects) can be estimated by maximum likelihood, BLUP.
- Natural decomposition of the vector field

Future Works

- Improve computational efficiency to handle large amount of data
- Extension the model to incorporate radial component to deal with data on spherical shell (different r).

References I

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Thank You!

atykim@ucdavis.edu

Inner Product

- Given two complex-valued functions f, g, define their inner product to be
$$\langle f, g\rangle = \int_{\mathbb{S}^2} f g^* d\mu$$
where g^* is conjugate of g
- Given two tangential vector fields $\mathbf{u}, \mathbf{v}$, define their inner product to be
$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_{\mathbb{S}^2} u^\theta v^{\theta *} + u^\phi v^{\phi *} d\mu$$
where $\mathbf{v} = v^\theta \hat{\boldsymbol{\theta}} + v^\phi \hat{\boldsymbol{\phi}}$

where $d\mu = \sin\theta d\theta d\phi$.

Scalar Spherical Harmonics

For every integer $l = 0, 1, 2, \ldots,$ and every $m = -l, \ldots, l$, the complex **Scalar spherical harmonic function** $Y_{l,m}$ at (θ, ϕ), $0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$ is

$$
\begin{aligned}
Y_{l,m}(\theta,\phi) &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\theta)\exp(im\phi), && m \geq 0 \\
Y_{l,m}(\theta,\phi) &= (-1)^m Y^*_{l,-m}(\theta,\phi), && m < 0
\end{aligned}
$$

where $\{P_{l,m}\}$ denotes the *associated Legendre function*.

Some properties of SSH

- Orthogonal system of functions on a surface sphere $\mathbb{S}^2$: The spherical harmonics define a complete basis over the sphere.
- Basis functions in $L^2(\mathbb{S}^2)$ with the properties
  - Normalized: $\|Y_{l,m}\| = 1$
  - Orthogonal: $\langle Y_{l,m}, Y_{l',m'} \rangle = \int_0^{\pi} \int_0^{2\pi} Y_{l,m} Y^*_{l',m'} \sin\theta d\phi d\theta = \delta_{ll'}\delta_{mm'}$
  - Completeness: Any function h in $L^2(\mathbb{S}^2)$ can be written a linear combination of $Y_{l,m}$:
    $h(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{l,m} Y_{l,m}(\theta,\phi)$,
    where $f_{l,m} = \langle h, Y_{l,m} \rangle$ note that $f_{l,-m} = (-1)^m f^*_{l,m}$

VSH: Properties

For $\{\mathbf{Y}_{l,m}, \mathbf{B}_{l,m}, \mathbf{C}_{l,m}\}$,

- Parity: $\mathbf{Y}_{l,-m} = (-1)^m \mathbf{Y}^*_{l,m}, \mathbf{B}_{l,-m} = (-1)^m \mathbf{B}^*_{l,m}, \mathbf{C}_{l,-m} = (-1)^m \mathbf{C}^*_{l,m}$
- Orthogonality:
  - $\mathbf{Y}_{l,m} \cdot \mathbf{B}_{l,m} = 0, \mathbf{Y}_{l,m} \cdot \mathbf{C}_{l,m} = 0, \mathbf{B}_{l,m} \cdot \mathbf{C}_{l,m} = 0$
  - $\langle \mathbf{Y}_{l,m}, \mathbf{Y}_{l',m'} \rangle = \delta_{ll'}\delta_{mm'}, \langle \mathbf{B}_{l,m}, \mathbf{B}_{l',m'} \rangle = \delta_{ll'}\delta_{mm'}, \langle \mathbf{C}_{l,m}, \mathbf{C}_{l',m'} \rangle = \delta_{ll'}\delta_{mm'}$
- A complete orthogonal basis (spherical coordinate vector basis) on $\mathbb{S}^2$, any vector field $\mathbf{v}$ on $\mathbb{S}^2$ can be decomposed as

$$\mathbf{v}(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(f^r_{l,m} \mathbf{Y}_{l,m} + f^B_{l,m} \mathbf{B}_{l,m} + f^C_{l,m} \mathbf{C}_{l,m} \right)$$

where $f^r_{l,m} = \langle \mathbf{v}, \mathbf{Y}_{l,m} \rangle, f^B_{l,m} = \langle \mathbf{v}, \mathbf{B}_{l,m} \rangle, f^C_{l,m} = \langle \mathbf{v}, \mathbf{C}_{l,m} \rangle$

Complex Gaussian Random Variables

$f^B_{l,m} \sim CN(0, \sigma^2_{B,l}), f^C_{l,m} \sim CN(0, \sigma^2_{C,l})$:

$$\begin{cases} \Re(f^B_{lm}) \sim N(0, \frac{1}{2}\sigma^2_{B,l}) \\ \Re(f^B_{lm}) \sim N(0, \sigma^2_{B,l}) \\ \Re(f^B_{l,m}) = (-1)^{|m|}\Re({f^B_{l,-m}}^*) \end{cases} \quad \begin{cases} \Im(f^B_{lm}) \sim N(0, \frac{1}{2}\sigma^2_{B,l}) & m > 0 \\ \Im(f^B_{lm}) = 0 & m = 0 \\ \Im(f^B_{l,m}) = (-1)^{|m|+1}\Im({f^B_{l,-m}}^*) & m < 0 \end{cases}$$

where $f_{l,m} = \Re(f_{l,m}) + i\Im(f_{l,m})$.

- The coefficients are independent across l
- For the fixed l, $\Re(f^B_{l,m}), \Re(f^C_{l,m})$, $\Im(f^B_{l,m}), \Im(f^C_{l,m})$ are independent identically distributed across m.
- $\sigma^2_{B,l} = l^{-\alpha_B}\sigma^2_B$, $\sigma^2_{C,l} = l^{-\alpha_C}\sigma^2_C$

Log-Likelihood

A tangential vector random fields $\mathbf{y}_t = \big(y_t(s_1), y_t(s_2), \cdots, y_t(s_{n_t})\big)$ at locations s_i, $i = 1, \cdots, n_t$, and time t, $t = 1, \cdots, k$
$\mathbf{y}_t \sim N(\mathbf{0}, V_t(\boldsymbol{\theta}))$ where $\boldsymbol{\theta} = [\sigma_B^2, \sigma_C^2, \tau^2]$.
The log-likelihood is

$$\ell(\boldsymbol{\theta}) = -\frac{\sum_{t=1}^{k} n_t}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{k}\Big[\log(|V_t(\boldsymbol{\theta})|) + (\mathbf{y}_t^T V_t(\boldsymbol{\theta})^{-1}\mathbf{y}_t)\Big] \quad (6)$$

where $V_t(\boldsymbol{\theta}) = \mathbf{Z_t} G \mathbf{Z_t}^T + \tau^2 I_{2n_t}$