Recovery of ruin probability and Value at Risk from the scaled Laplace transform inversion

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Abstract

We propose three modified approximations of the the ruin probability and its inverse function using the inversion of the scaled values of Laplace transform suggested by Mnatsakanov et al. (2015).

We consider

- the classical risk model for the evaluation of the probability of ultimate ruin and its inverse function,
- The problem of evaluating numerically the tail Value at Risk of an insurance portfolio will be discussed briefly,
- Performances of the proposed constructions are demonstrated via graphs and tables using several examples.

Outline

- **Introduction**
- **Ruin probability**
- **Inverse Function of the Ruin Probability and Value at Risk**
- **Simulation results**

Scaled Laplace transform

Assume that F is an aboslutely continuous distribution with $\mathbb{R}^+ = [0, \infty)$ as its support. Let f be the probability density function of F with respect to the Lebesgue measure on $\mathbb{R}^+$.

Definition

Suppose that a random variable X is distributed according to F. Assume also that we have a given sequence $\mu(F) = \{\mu_t(F), t \in \mathbb{N}_\alpha\}$. The scaled Laplace transform of F is defined by,

$$\mu_t(F) := \mathcal{L}_{F,b}(t) = \int_{\mathbb{R}_+} e^{-ctx} dF(x) \quad \text{for} \quad t \in \mathbb{N}_\alpha = \{0, 1, \ldots, \alpha\}, \alpha = 1, 2, \ldots. \tag{1}$$

where, $c = \ln b$, for some $b > 1$.

Inversion formula

From Mnatsakanov et al. [6], the cumulative distribution function F and probability density function f, are approximated by:

$$F_{\alpha,b}(x) := (\mathcal{L}_{\alpha,b}^{-1}\mu(F))(x) = 1 - \sum_{k=0}^{[\alpha b^{-x}]} \sum_{j=k}^{\alpha} \binom{\alpha}{j} \binom{j}{k} (-1)^{j-k} \mu_j(F)$$

and

$$f_{\alpha,b}(x) := \frac{[\alpha b^{-x}] \ln(b) \Gamma(\alpha+2)}{\alpha \Gamma([\alpha b^{-x}]+1)} \sum_{j=0}^{\alpha-[\alpha b^{-x}]} \frac{(-1)^j \mu_{j+[\alpha b^{-x}]}(F)}{j!(\alpha - [\alpha b^{-x}] - j)!}, \quad x \in \mathbb{R}^+ \tag{2}$$

$\alpha \in \mathbb{N}$ is an integer-valued parameter.

From [5, 6], we see that $F_{\alpha,b}$ and $f_{\alpha,b}$ converge uniformly to F and f respectively, as $\alpha \to \infty$.

Classical Risk Model

- A risk reserve process $\{R_t\}_{t\geq 0}$ is a model for the time evolution of the reserves of an insurance company.
- Let $R_0 = u > 0$ be the initial reserve at time $t = 0$.
- The company receives income from premiums at a constant rate say, p, per unit time.
- Claims are paid according to a compound process $S(t) = \sum_{k=1}^{N(t)} X_k$.
- $\{N(t), t \geq 0\}$, the total number of claims is a Poisson process with intensity $\lambda > 0$ and the individual claims $X_1, X_2, \ldots$, are i.i.d. nonnegeative random variables, independent of $N(t)$, see [2, 3].
- At time t, the reserve of the company is $R_t = u + pt - S(t)$ and the time to ruin is $\tau(u) = \inf\{t \geq 0 : R_t < 0\}$.

Properties of the risk model

- The classical risk model has the property that there exists a constant ρ such that

$$\frac{1}{t}\sum_{k=1}^{N(t)} X_k \rightarrow_{a.s.} \rho, \ t \rightarrow \infty.$$

- ρ is defined as the average amount of claims per unit time.
- Another basic quantity is the **safety loading** (or the **security loading**) η. It is defined as the relative amount by which the premium rate p must exceed ρ,

$$\eta = \frac{p-\rho}{\rho}.$$

- An insurance company must ensure that $\eta > 0$ **always**, to avoid certain ruin.

Ruin probability in finite and infinite time horizons

- The probability $\psi(u)$ of ultimate ruin is the probability that the reserve ever drops below zero,

$$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right) = \mathbb{P}(\tau(u) < \infty). \tag{3}$$

- The probability that ruin occurs before time T is

$$\psi(u,T) = \mathbb{P}\left(\inf_{0\leq t\leq T} R_t < 0 | R_0 = u\right) = \mathbb{P}(\tau(u) < T). \tag{4}$$

- $\psi(u)$ and $\psi(u,T)$ are referred to as ruin probabilities with inifinite horizon and finite horizon, respectively.

Moment Recovered Approximation

- The Laplace transform of the ruin probiability according to the Pollaczek-Khinchine formula has the form, we obtain the scaled Laplace transform of $\psi(u)$ as:

$$\mathcal{L}_{\psi}(s\ln b) = \frac{1}{s\ln b} - \frac{1-\rho}{s\ln b - \lambda_p(1-\mathcal{L}_f(s\ln b))} \tag{5}$$

where, $\mathcal{L}_g(s\ln b) = \int_{\mathbb{R}^+} e^{-s\ln b} g(x)\mathrm{d}x$ is the scaled Laplace transform of some function g defined on $\mathbb{R}^+ = [0,\infty)$, $\rho = \lambda_p E(X) < 1$ and $\lambda_p = \lambda/p$.

The basic approximation of $\psi(u)$ based on using the sequence of operators $\{\mathcal{L}_{\alpha}^{-1}, \alpha \in \mathbb{N}_+\}$ is defined as:

$$\psi^*_{\alpha,b}(u) := (\mathcal{L}^{\alpha}_{-1}\mathcal{L}_{\psi})(u) = \frac{[\alpha b^{-u}]\ln b\Gamma(\alpha+2)}{\alpha\Gamma([\alpha b^{-u}]+1)} \sum_{m=0}^{\alpha-[\alpha b^{-u}]} \frac{(-1)^m\mathcal{L}_{\psi}((m+[\alpha b^{-u}])\ln b)}{m!(\alpha-[\alpha b^{-u}]-m)!} \tag{6}$$

Rate of Approximation

Theorem

As $\alpha \to \infty$, we have

$$\psi_{\alpha,b}(x) - \psi(x) = \frac{1}{\alpha+1}\left[-\frac{b^x\psi'(x)}{\ln b}([\alpha\phi(x)] - \alpha\phi(x) - \phi(x)) + \frac{1}{2}\left(\frac{b^{2x}\psi'(x)}{\ln b} + \frac{b^{2x}\psi''(x)}{\ln^2 b}\right)\phi(x)(1-\phi(x))\right] + o\left(\frac{1}{\alpha}\right) \quad (7)$$

Modified MR-approximations

- Let $\tilde{\alpha} = 2\alpha$
- The modified MR-approximation of ψ is defined as

$$\widetilde{\psi}_{\alpha,b}(x) := 2\psi_{\tilde{\alpha},b}(x) - \psi_{\alpha,b}(x) \tag{8}$$

Corollary

Assume ψ is continuous. Then for each $b > 1$, $\widetilde{\psi}_{\alpha,b}$ converges uniformly to ψ and

$$\begin{aligned}\widetilde{\psi}_{\alpha,b}(x) - \psi(x) = \frac{1}{(2\alpha+1)(\alpha+1)}\Bigg[\psi(x) \\ - \frac{\psi'(x)b^x}{\ln b}(2\alpha+3)([\alpha\phi(x)] - \alpha\phi(x) - \phi(x)) \\ + \left(\frac{\psi'(x)b^{2x}}{\ln b} + \frac{\psi''(x)b^{2x}}{\ln^2 b}\right)\phi(x)(1-\phi(x))\Bigg] + o\left(\frac{1}{\alpha^2}\right)\end{aligned} \tag{9}$$

as $\alpha \to \infty$.

Smooth Approximations

- Consider the Szász-Mirakayan operator $S_\alpha(\psi)$ defined as:

$$(S_\alpha(\psi))(x) = \sum_{k=0}^{\infty} \psi\left(\frac{k}{\alpha}\right) P_\alpha(k,x), \; x \in \mathbb{R}_+. \tag{10}$$

- We define smooth MR approximation of ψ as

$$\psi_{\alpha,P}(x) := (S_\alpha(\psi_{\alpha,b}))(x) \tag{11}$$

and

- the smooth version of $\widetilde{\psi}_{\alpha,b}$ as

$$\widetilde{\psi}_{\alpha,P}(x) := (S_\alpha(\widetilde{\psi}_{\alpha,b}))(x) \tag{12}$$

where, $x \in \mathbb{R}_+$, the wieghts $P_\alpha(k,x) = \frac{(\alpha x)^k}{k!} e^{-\alpha x}$, represent the Poission probabilities for $k \geq 0$.

Value-at-Risk

- **Value-at-Risk (VaR)** is an estimate of the **worst** possible monetary loss from a financial investment over a future time-period e.g. one-day, one-week, one-month, etc.
- It quantifies how much an economic agent can expect to lose in one day, week, year, . . . with a given probability.
- For a given time horizon, say t, and confidence level $p \in (0, 1)$, the VaR of a portfolio is the loss in **market value** over the time horizon t that is exceeded with probability $1 - p$.
- Consider the continuous-time risk model $\{R_t\}_{t \geq 0}$. Instead of fixing the initial capital u, we fix the safety loading η and ask for the amount of the initial capital needed to bound the ruin probability by an acceptable level, say ϵ.
- We then define a ruin-consistent VaR-type risk measure as:

$$VaR(\epsilon) = \inf\{u \geq 0 | \psi(u) \leq \epsilon\} = \psi^{-}(\epsilon).$$

Approximation of VaR

To approximate the VaR i.e. the inverse function of ψ^- of ruin probability, consider

$$\psi^-_{\alpha,b}(x) = \int_0^\infty \bar{B}_\alpha(\psi_{\alpha,b}(u), x)du \tag{13}$$

where $\bar{B}_\alpha(t,x) = 1 - B_\alpha(t,x)$ and

$$B_\alpha(t,x) = \sum_{k=0}^{[\alpha x]} \binom{\alpha}{k} t^k (1-t)^{\alpha-k} \to \begin{cases} 1, & t < x \\ 0, & t > x \end{cases} \quad \text{as } \alpha \to \infty \tag{14}$$

Approximation of VaR

Note that (13) can be expressed as

$$\psi^{-}_{\alpha,b}(x) = \sum_{l=1}^{\alpha} \int_{u_{l-1}}^{u_l} \bar{B}_{\alpha}(\psi_{\alpha,b}(u), x) \; dx = \sum_{l=1}^{\alpha} \bar{B}_{\alpha}(\psi_{\alpha,b}(u_{l-1}), x) \Delta u_l \tag{15}$$

where $u_l = [\ln \alpha - \ln(\alpha - j + 1)] / \ln b$ for $1 \leq l \leq \alpha$, $\Delta u_l = u_{l-1} - u_l$ and

$$\psi_{\alpha,b}(u_l) = \frac{\ln(b)\Gamma(\alpha+1)}{\Gamma(\alpha - l + 2)} \sum_{j=0}^{l-1} \frac{(-1)^j \mathcal{L}_{\psi}((j + \alpha - l + 1)\ln(b))}{j!(l-1-j)!}. \tag{16}$$

- When there is no explicit expression of ψ^{-}, it is approximated by

$$\psi^{-}_{\alpha^*}(x) = \int_0^{\infty} \bar{B}_{\alpha^*}(\psi(u), x) \mathrm{du}$$

Smooth Approximation of ψ^-

Consider the Bernstein approximation (of order α) of some continuous function f defined on $[0, 1]$:

$$B^*_\alpha(f, x) = \sum_{k=0}^{\alpha} f\left(\frac{k}{\alpha}\right)\binom{\alpha}{k} x^k(1-x)^{\alpha-k} \tag{17}$$

Applying (13) and (17), we construct polynomial approximation of $\psi^-(x)$ as follows: let

$$b_\alpha(k, x) = \binom{\alpha}{k} x^k(1-x)^{\alpha-k} \text{ for } k = 0, 1, \cdots, \alpha.$$

Then,

$$\psi^{-1}_{\alpha,B}(x) := (\psi^{-1}_{\alpha,b} \bullet b_\alpha)(x) = \sum_{k=0}^{\alpha} \psi^{-1}_{\alpha,b}\left(\frac{k}{\alpha}\right) b_\alpha(k, x) \tag{18}$$

is the smoothed MR-approximation of $\psi^-(u)$.

Simulation Results

- Explicit expressions of $\psi(u)$ and $\psi^{-}(u)$ are generally difficult to obtain.
- Only $\psi(0) = \lambda E(X_1)/p$ is known.
- Comparison is done with the Fixed Talbot method studies by Albrecher et al. [1]
- All evaluations of $\psi_{\alpha,b}(u)$ and $\widetilde{\psi}_{\alpha,b}(u)$ were conducted at

$$u \in \{\ln(\alpha/(\alpha - j + 1))/\ln b, 1 \leq j \leq \alpha\}.$$

Empirical Estimations

- When the distrbution F of X is unknown but the sample $X_1, X_2, \ldots, X_n$ from F is available
- Considering $\overline{X}$, $\widehat{\rho} = \lambda_p \overline{X}$ and the empirical Laplace transform of F

$$\widehat{\mathcal{L}}_f(s) = \frac{1}{n} \sum_{i=1}^{n} e^{(-\ln b)X_i}$$

- We estimate $\psi(u)$ by

$$\widehat{\psi}_{\alpha,b}(u) = \frac{\alpha}{\alpha+1}(\mathcal{L}_{\alpha}^{-1}\widehat{\mathcal{L}}_{\psi}) \tag{19}$$

- The modfied version of $\widehat{\psi}$ is defined as

$$\widetilde{\widehat{\psi}}_{\alpha,b}(u) = 2\widehat{\psi}_{\tilde{\alpha},b}(u) - \widehat{\psi}_{\alpha,b}(u)$$

Example: $X \sim \text{Gamma}(\text{shape} = a, \text{scale} = \beta)$

- Consider a surplus model with claims sizes specified by Gamma(2, 1), $\lambda = 1$, $p = 5$.
- Gyzl et al.[4] showed that $\psi(u) = 0.461862e^{-0.441742u} - 0.061862e^{-1.358257u}$ for $u > 0$.

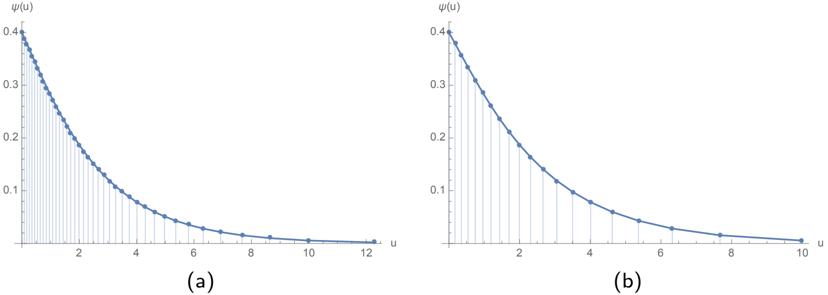


Figure: Approximations $\psi(u)$. $X \sim \text{Gamma}(2,\ 1)$ by (a) $\psi_{\alpha,b}$ (dots), $\alpha = 40$, and (b) $\tilde{\psi}_{\alpha,b}$ (dots), $\alpha = 20$. In both plots, $b = 1.35$.

Smooth Approximations

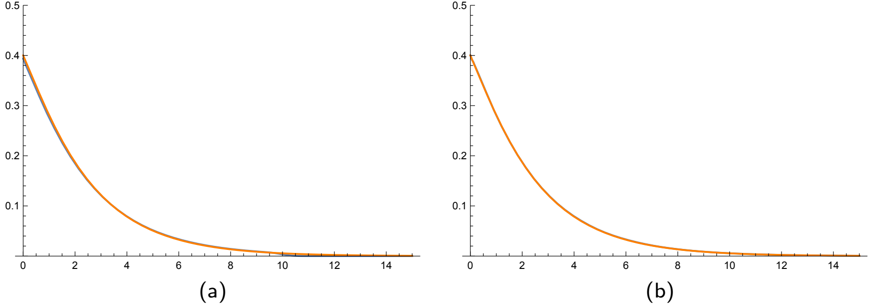
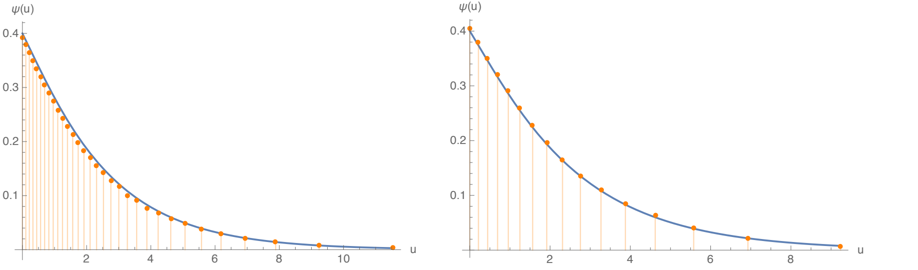


Figure: Let $X \sim \text{Gamma}(2, 1)$. Approximation of $\psi(u)$ (orange curve) by (a) $\psi_{\alpha,P}(u)$,$\alpha = 100$, and (b) $\tilde{\psi}_{\alpha,P}(u)$ when $\alpha = 40$. In both plots $b = 1.125$ and $n = 1000$.

Estimation



(a) (b)

Figure: Estimation of $\psi(u)$ (blue curve), when $X \sim \text{Gamma}(2,1)$, (a) $\widehat{\psi}_{\alpha,b}$ (dots), $\alpha = 40$ (b) $\widetilde{\widehat{\psi}}_{\alpha,b}$ (dots), $\alpha = 20$. In both plots, $b = 1.35$ and $n = 1000$.

Approximation error

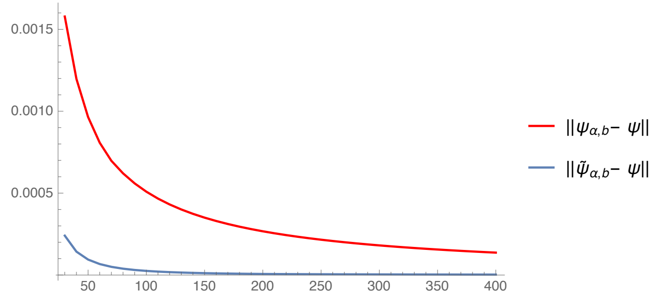


Figure: $X \sim$ Gamma$(2, 1)$. Approximation errors in sup-norm for $\alpha = 10k, 3 \leq k \leq 40$, $1.35 \leq b \leq 1.50$.

Normalized Errors

Table: Records of $10^4 \times ||\widetilde{\psi}_{\alpha,b} - \psi||$ with $X \sim \text{Gamma}(2,1)$ and $\lambda_p = 0.2$.

$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
30	2.77275	2.49112	2.44116	2.4169	2.40494	2.3931	2.36977	2.34688	2.25958	2.06786
45	1.35393	1.19594	1.16873	1.15559	1.14914	1.14276	1.13021	1.11796	1.07158	0.971673
60	0.798906	0.699685	0.682807	0.674683	0.670697	0.666759	0.659028	0.651483	0.623045	0.562313
75	0.526332	0.458597	0.447158	0.441659	0.438963	0.436302	0.431079	0.425987	0.406834	0.366135
100	0.30483	0.264232	0.257423	0.254154	0.252553	0.250972	0.247874	0.244855	0.233523	0.20956
200	0.0796509	0.0685068	0.0666558	0.0657692	0.0653352	0.0654537	0.0663055	0.0669547	0.0673546	0.0537883

Table: Records of $10^4 \times ||\psi_{\alpha,b} - \psi||$ with $X \sim \text{Gamma}(2,1)$ and $\lambda_p = 0.2$.

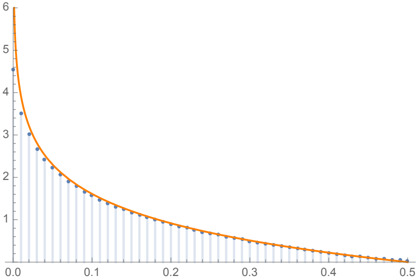
$\alpha \setminus b$	1.35	1.40	1.41	1.415	1.4175	1.42	1.425	1.43	1.45	1.50
60	15.398	9.64228	8.72093	8.291	8.07721	8.24543	8.82845	9.39798	11.5503	16.1823
90	10.3284	6.47838	5.86249	5.57266	5.59349	5.78961	6.17496	6.5514	7.97399	11.0357
120	7.77521	4.87904	4.41573	4.19713	4.3097	4.45609	4.74374	5.02473	6.08666	8.3723
150	6.2329	3.91266	3.54197	3.38572	3.50388	3.62064	3.85007	4.07419	4.92122	6.7444
200	4.68474	2.94233	2.66394	2.5823	2.67064	2.75794	2.92948	3.09705	3.73036	5.09364
400	2.35087	1.47804	1.33861	1.32403	1.36799	1.41142	1.49677	1.58015	1.89528	2.57371

$X \sim \text{Gamma}(2, 1)$

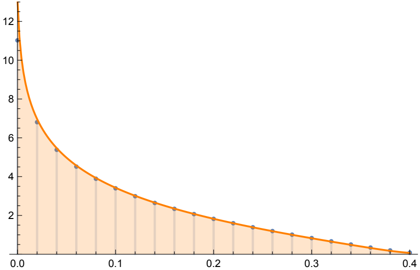
Table: Values of ultimate ruin probabilities $\widetilde{\psi}_{\alpha,b}$, $\psi_{\alpha,b}$ and ψ_{FT} computed at several values of $u_j = \ln(\alpha/(\alpha - j + 1))/\ln b$. $X \sim \text{Gamma}(2,\ 1)$, $\alpha = 5000$, $b = 1.425$, $\lambda = 1$ and $p = 5$.

j	500	600	700	800	900	1000
$\psi(u_j)$	0.362835	0.354857	0.346728	0.338460	0.330063	0.321547
$\widetilde{\psi}_{\alpha,b}(u_j)$	0.362835	0.354857	0.346728	0.338460	0.330062	0.321546
$\psi_{\alpha,b}(u_j)$	0.362832	0.354853	0.346723	0.338453	0.330055	0.321538
$\psi_{FT}(u_j)$	0.362834	0.354856	0.346728	0.338460	0.330062	0.321546
j	2000	2500	3000	3500	4000	4500
$\psi(u_j)$	0.232093	0.186355	0.141464	0.098557	0.058914	0.024346
$\widetilde{\psi}_{\alpha,b}(x_j)$	0.232093	0.186355	0.141464	0.098557	0.058914	0.024346
$\psi_{\alpha,b}(u_j)$	0.232084	0.186349	0.141461	0.098559	0.058919	0.024352
$\psi_{FT}(u_j)$	0.232092	0.186355	0.141463	0.098556	0.058913	0.024345

Approximation of ψ^-



(a)



(b)

Figure: Approximation of ψ^- (orange) by $\psi_{\alpha,b}$ (dots) (a) $X \sim \text{Exp}(2)$, $\psi(x) = \frac{1}{2}e^{-x}$, $\alpha = 100$, $b = 1.4125$; (b) when $\psi(x) = 0.461862e^{-0.441742x} - 0.061862e^{-1.358257x}$, $\alpha = 50$, $b = 1.4125$, The true ψ^- is replaced by $\psi^-_{\alpha*}$, with $\alpha^* = 500$

Tail-VaR

- Trufin et al. [7] defined the Tail-VaR as:

$$\mathrm{TVaR}(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \psi^{-1}(x)dx, \text{ for } \epsilon \in [0,\ 1].$$

- From $\psi_{\alpha,b}^{-1}$, we constructed the MR-approximation of $\mathrm{TVaR}(\epsilon)$ as

$$\mathrm{TVaR}_{\alpha,b}(\epsilon) := \frac{1}{\epsilon}\int_0^{\epsilon} \psi_{\alpha,b}^{-1}(x)dx = \frac{1}{\alpha\epsilon}\sum_{l=1}^{[\alpha\epsilon]} \psi_{\alpha,b}^{-1}\left(\frac{l-1}{\alpha}\right) \tag{20}$$

Average VaR

Table: Values of Tail-VaR. $X \sim \text{Exp}(1/5)$, $\alpha = 200$, $b = 1.05$

ϵ	$\text{TVaR}_{\alpha,b}(\epsilon)$	$\text{TVaR}(\epsilon)$
0.80	30.7981	31.2247
0.90	27.4187	27.6912
0.95	25.9756	26.0692
0.99	24.9261	25.9756

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Thank you!