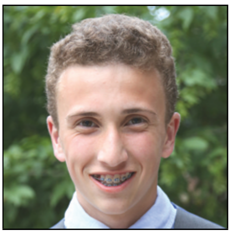
Challenging Magic Squares for Magicians

Arthur T. Benjamin and Ethan J. Brown

Arthur T. Benjamin (benjamin@math.hmc.edu) has taught at Harvey Mudd College since 1989, after earning his Ph.D. from Johns Hopkins in mathematical sciences. He has served as editor of *Math Horizons* and the Spectrum Book Series, has written two books for the MAA, and was a Pólya Lecturer from 2006 to 2008. He is excited that the 2014 theme for Mathematics Awareness Month is "Mathematics, Magic, and Mystery."



Ethan J. Brown (ebrown@andover.edu) is a fourteen-year-old freshman at Phillips Academy Andover in Andover MA. He enjoys soccer, track, mock trial, improvisational comedy, and of course, mathematics. In 2012, he established a new world record by memorizing and reciting 2012 digits of tau (6.283185...) as a fundraiser for his town library. He presents the magic square as the finale of his mental math stage show.

Magic squares are fascinating, and not just for mathematicians. Most people are intrigued to see numbers arranged in a box where every row, column, and diagonal have the same magical sum. Four-by-four magic squares, like the one shown in Figure 1, are especially intriguing because so many sets of four entries have the same magic total. Notice how every row, column, diagonal, broken diagonal, 2×2 box, and more, add up to 34.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Figure 1. A magic square with magic total 34.

Magic squares with a given total

Many magicians, including the authors of this paper, create magic squares as parts of their shows. Typically, an audience member is asked for a number (say between 30 and 100) and the magician quickly creates a magic square and shows off the many ways that their total is obtained. As described in many magic books (such as [**5**]), the

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8	11	$T-20$	1
$T-21$	2	7	12
3	$T-18$	9	6
10	5	4	$T-19$

Figure 2. A quick magic square with magic total T.

quickest and easiest way to create a magic square with total T is to modify the square in Figure 1 to create the square in Figure 2.

That construction has the advantage that it is very easy to create, since twelve of the sixteen numbers are always in the same location, subtracting 20 from the magic total easily provides the first of the four other numbers, and the other three numbers are within 1 or 2 of that. The disadvantage of this method is that with a large total, four of the numbers are conspicuously larger than the others. This could result in a magic square that is less aesthetically pleasing and potentially easier for the audience to figure out and determine the secret of its construction. See Figure 3(a) for an example.

(a)

8	11	56	1
55	2	7	12
3	58	9	6
10	5	4	57

(b)

18	21	26	11
25	12	17	22
13	28	19	16
20	15	14	27

Figure 3. Two magic squares with total 76.

We prefer the following algorithm, described in [**2, 4**], where all the numbers in the magic square are in near-consecutive order. Suppose that $T = 34 + 4q$. Then, by simply adding q to each square in Figure 1, you obtain a magic square of total T. Naturally, the numbers are still consecutive (from $1 + q$ to $16 + q$) and every group of 4 squares that add to 34 in Figure 1 now adds to T in the new square. If $T = 34 + 4q + r$ (where $r = 1, 2,$ or 3), then we add q to each square, but also add an additional r to the squares occupied by 13, 14, 15, and 16 (conveniently located in different rows, columns, and diagonals). For example, with a magic total of 76, then $q = 10$ and $r = 2$, so starting with the square in Figure 1, you add 10 to every square, except for the four largest numbers for which we add 12. See Figure 3(b). The numbers in the resulting magic square have no repeated numbers, are nearly consecutive, and preserve most, but not all, of the symmetries of the original magic square. If the magic square in Figure 1 is memorized, then this magic square can also be created quite quickly. We think most people would agree that the magic square created in Figure 3(b) is more satisfying than the one in Figure 3(a).

Magic squares for a given birthday

To give the magic square a more personal touch and to increase the perceived difficulty of construction, the magician can ask for the volunteer's birthday, and write it in the first row of the magic square. The construction in Figure 4, for the birthday $A/B/CD$ has dozens of groups of four squares with magic total $A + B + C + D$. We call this Construction 1. Notice that all the expressions in Construction 1 are different, so the resulting magic square tends to have very few repeated numbers.

(a)

A	B	C	D
$C+3$	$D-3$	$A+3$	$B-3$
$D-2$	$C-2$	$B+2$	$A+2$
$B-1$	$A+5$	$D-5$	$C+1$

(b)

3	**19**	**6**	**1**
9	−2	6	16
−1	4	21	5
18	8	−4	7

Figure 4. Construction 1 for a birthday magic square.

An alternative method is given by Construction 2, presented in Figure 5, which has a "surprise ending" that the spectator's birthday also appears in the four corners of the square. Notice that the construction works for any value of x. Naturally, choosing $x = 0$ produces a (not very magical looking) Latin square. In practice, we usually choose $x = 1$ (but see the performance tips in the last section). Although this square is easy to construct, it has the aesthetic drawback that some of the numbers are repeated. Not only do the bottom corner squares (intentionally) duplicate two of the numbers in the top row, but some of the numbers in the second and third rows are duplicates as well.

(a)

A	B	C	D
$C-x$	$D+x$	$A-x$	$B+x$
$D+x$	$C+x$	$B-x$	$A-x$
B	$A-2x$	$D+2x$	C

(b)

6	**20**	**9**	**9**
8	10	5	21
10	10	19	5
20	4	11	9

Figure 5. Construction 2 for a double birthday magic square.

Magic squares with three random digits

For the ultimate challenge, the authors thought it would be especially impressive to allow members of the audience to choose the total and three numbers to be placed in any three of the squares. (Naturally, asking the audience for four numbers and the total could lead to an impossible problem.) The solution, as performed by the second author, takes advantage of previous Constructions 1 and 2.

Ask members of the audience to choose any three squares, then ask for any three numbers to go inside those squares (say between 1 and 20). Finally, ask another audience member to provide the total (say between 30 and 80). Note that the first three squares can be chosen in $\binom{16}{3} = 560$ ways. In $4^4 = 256$ of those situations, the three chosen locations will correspond to three different letters from Figure 4(a), say using letters A, B, and C.

For example, if the chosen numbers are 3, 11, and 13 in the squares prescribed in Figure 6(a), then Construction 1 provides a solution by letting $A = 3$, $B = 14$, and $C = 15$. If the prescribed total is $T = 41$, then that would force $D = T - (A + B + C) = 9$, and the square could be completed as in Figure 6(b).

(a)

3			
			11
	13		

(b)

3	14	15	9
18	6	6	**11**
7	**13**	16	5
13	8	4	16

Figure 6. Constructing a magic square with total 41 with three prescribed squares.

If the three locations do not correspond to different letters in Figure 5(a), then they often correspond to different letters when Figure 5(a) is turned counterclockwise 90 degrees, as displayed in Figure 7(a). For example, in Figure 7(b), the prescribed numbers 6, 5, and 18 all use B numbers from Figure 5(a), but they correspond to $A = 6$, $B = 8$, and $D = 23$ in Figure 7(a). So with a prescribed total of 49, the magician need only "tilt his or her head" and apply the same process of Construction 1 to reach the completed square in Figure 7(b). We call this process Construction 1R.

(a)

D	$B-3$	$A+2$	$C+1$
C	$A+3$	$B+2$	$D-5$
B	$D-3$	$C-2$	$A+5$
A	$C+3$	$D-2$	$B-1$

(b)

23	**5**	8	13
12	9	10	**18**
8	20	10	11
6	15	21	7

Figure 7. Constructing a magic square with total 49 with three prescribed squares using Construction 1R.

Conveniently, in the $4^2 = 16$ situations when all three prescribed squares use the same letter in Figure 4(a), those squares will correspond to different letters in the square of Figure 7(a). (Indeed, that would be true even if there were four prescribed squares with the same letter.) Moreover, $4 \cdot 6 \cdot 6 = 144$ situations are of the form YYZ (with $Z \neq Y$) in Figure 4(a), that correspond to three different letters in Figure 7(a). To see this, there are four choices for Y, then $\binom{4}{2} = 6$ ways to choose which two squares in Figure 5(a) to use with that letter. Those will correspond to two different letters in Figure 7(a), say A and B. Then there are $8 - 2 = 6$ choices for the remaining square corresponding to a C or D in Figure 7(a) that is not one of the remaining Y's of Figure 4(a). Thus using Construction 1 or 1R, we can handle $256 + 16 + 144 = 416$ of the 560 ways of choosing three locations for the prescribed numbers. Note that if two locations were prescribed, instead of three, then Construction 1 or 1R handles all $\binom{16}{2} = 120$ possibilities.

Combinatorial aside. The letters in Figure 4(a) and its rotation are orthogonal Latin squares, in that their combined letters give all 16 possible ordered pairs, as shown in Figure 8. This allows us to count the number of ways to choose three squares that

AD	*BB*	*CA*	*DC*
CC	*DA*	*AB*	*BD*
DB	*CD*	*BC*	*AA*
BA	*AC*	*DD*	*CB*

Figure 8. The letters of Figures 4(a) and 7(a) are orthogonal Latin squares.

will correspond to different letters in both matrices. The *first* square can be chosen 16 ways, then the *second* square can be chosen $3^2 = 9$ ways (three choices for the first coordinate, then three choices for the second coordinate), then the *third* square can be chosen $2^2 = 4$ ways. Since order does not matter, there are $(16 \cdot 9 \cdot 4)/3! = 96$ ways to select three squares that will satisfy both conditions. Hence, by the Principle of Inclusion-Exclusion, the number that satisfies at least one condition is $256 + 256 - 96 = 416$, as previously noted.

We are left with with $560 - 416 = 144$ situations like the one in Figure 9. This can also be counted directly. To create a *YYZ* in Figures 4(a) and 7(a), there are four choices for Y to be used in Figure 4(a), and then $\binom{4}{2} = 6$ ways to pick their locations. These will necessarily correspond to different letters in Figure 7(a), say A and B, so there are $8 - 2 = 6$ ways to pick the other A or B square in Figure 7(a). This example can be extended to a magic square neither by Construction 1 (since it's of type *AAB*) nor Construction 1R (since its of type *BBD*). When this happens, we resort to "plan x."

8	**17**		
		5	

Figure 9. A square that foils Constructions 1 and 1R.

When Constructions 1 and 1R fail us, we go back to the birthday magic square of Figure 5(a), which we repeat in Figure 10(a). This has the same letter pattern as Construction 1, but it also has a variable parameter x that can be exploited. For example, in the example of Figure 9, we can assign $A = 8$, $B = 17$, and $x = 3$. We still have two degrees of freedom, so after the total is assigned, say $T = 52$, we can arbitrarily choose two numbers that add to $T - (A + B) = 27$, say $C = 21$ and $D = 6$, and the result is the magic square of Figure 10(b). We call this process Construction 2.

This process successfully handles $4 \cdot 4 \cdot 6 = 96$ of the 144 situations. For these *YYZ* situations, there are four choices for Y, then $\binom{4}{2} - 2 = 4$ choices for which pair we can use. (For example, with $Y = A$, we disallow picking A with $A - 2x$ or picking $A - x$ twice.) Then the third square can be chosen six ways to form *YYZ* in the rotated square. This can result in the occasional unaesthetic appearance of negative numbers.

(a)

A	B	C	D
$C-x$	$D+x$	$A-x$	$B+x$
$D+x$	$C+x$	$B-x$	$A-x$
B	$A-2x$	$D+2x$	C

(b)

8	**17**	21	6
18	9	**5**	20
9	24	14	5
17	2	12	21

Figure 10. "Completing the square" with Construction 2.

(For instance, if in the last example, the number 17 was replaced by 1, then the third number on the main diagonal would be $B - x = -2$.)

Of the remaining 48 situations, $4 \cdot 1 \cdot 6 = 24$ of them use two prescribed squares in Construction 2 with a difference of $2x$:

- A and $A - 2x$,
- $B + x$ and $B - x$,
- $C - x$ and $C + x$,
- D and $D + 2x$.

In each case, six choices of the third square will not be satisfied by Construction 1R. For example, suppose that the boxes $C - x$, $C + x$, and $B + x$ are prescribed, as in Figure 11(a).

(a)

A	B	C	D
$\boldsymbol{C-x}$	$D+x$	$A-x$	$\boldsymbol{B+x}$
$D+x$	$\boldsymbol{C+x}$	$B-x$	$A-x$
B	$A-2x$	$D+2x$	C

(b)

30	8	9	25
3	31	24	**14**
31	**15**	2	24
8	18	37	9

Figure 11. An "even" more challenging situation.

In order to apply Construction 2, the squares containing $C - x$ and $C + x$ must differ by an even number. (While we are willing to put up with negative numbers, we do not tolerate half-integers!) The simplest remedy is to restrict the parity of one of the entries. For example, if your volunteer chooses $C - x$ to be 3 and $B + x$ to be 14, then indicates square $C + x$, the magician can say, "To make this really interesting, give me any odd number between 1 and 20." (A more complex remedy without the parity restriction will be given in the final section.) Say the volunteer chooses $C + x = 15$, then $C = 9$, $x = 6$, and $B = 8$. If the prescribed total is 72 and you freely choose $A = 30$ and $D = 72 - (30 + 8 + 9) = 25$, then the square can be completed as in Figure 11(b).

The only remaining situation to consider, which occurs in just $4 \cdot 1 \cdot 6 = 24$ of the 560 possible cases, is when two of the prescribed squares are required to be equal by Construction 2. In other words, among the three prescribed squares, two have label $A - x$ or B or C or $D + x$, where the labels are given in Figure 11(a). Conveniently, in all 24 of these situations, when the magician mentally rotates the matrix 180 degrees, we obtain the previously considered case where the two duplicated numbers have a new letter, but differ by $2x$. Thus, the magician can tilt (or stand on) his or her head and complete the square using the previous process. For example, if the prescribed squares are chosen as in Figure 12(a), then the repeated B's, when rotated, become D and $D + 2x$, as in Figure 12(b). We call this process Construction 2R. Thus, between Constructions 1, 1R, 2, and 2R, we can, with just a little bit of practice, create a magic square based on any three prescribed locations with freely chosen numbers and magic total.

(a)

A	$\boldsymbol{B}$	C	D
$C - x$	$D + x$	$A - x$	$B + x$
$D + x$	$C + x$	$B - x$	$\boldsymbol{A - x}$
$\boldsymbol{B}$	$A - 2x$	$D + 2x$	C

(b)

C	$\boldsymbol{D + 2x}$	$A - 2x$	B
$A - x$	$B - x$	$C + x$	$D + x$
$B + x$	$A - x$	$D + x$	$\boldsymbol{C - x}$
$\boldsymbol{D}$	C	B	A

Figure 12. Construction 2 and its 180 degree rotation, Construction 2R.

Performance tips and extensions

Here are some ideas for presenting and creating magic squares for your audiences.

The magic square of Figure 5(a) can achieve even more ways to sum to the prescribed total by setting $x = (A + B - C - D)/2$, but that tends not to be worth the effort, even when x is an integer, since it adds to the mental effort and often generates more negative numbers in the square than is desirable. Normally, we just set $x = 1$ unless $A = 1$, since that will produce a negative number in the bottom row from $A - 2x = -1$. Thus, for January dates with $D \geq 2$, we choose $x = -1$ to avoid negative values. More performing tips for the double birthday magic square can be found in [**3**].

For all of the constructions with prescribed squares, it is best to fill out the rest of the magic square one letter at a time. The A's, B's, C's, and D's have the same geometrical pattern that allows them to be filled pretty quickly. Starting with the A in the corner, the other A expressions can be reached by a knight's move, then a diagonal move, then another knight's move. The same is true for the other letters as well. In fact, to give the appearance of an even greater challenge, after the audience member prescribes three squares, you can fill in three, six, or even nine more squares on your own before asking the audience member to give you the prescribed total.

For those who prefer mental calculation instead of memory, you can complete the square with practically nothing to memorize. For example, when doing the double birthday magic square of Figure 5(b), after the first row is given to you, write down the total $T = 44$ next to the magic square, then write $B = 20$ in the bottom left corner, write $C + x = 9 + 1 = 10$ in the square diagonally adjacent to that; then the rest of the squares are forced by groups of four that are required to sum to T. For instance, we now have three numbers on a diagonal, and since $20 + 10 + 9 = 39$, the missing number on that diagonal must be 5. Now, the upper right quadrant has three numbers

5, 9, and 9, so the missing number of that quadrant is 21. Then the top middle box has 20, 9, and 5, forcing the fourth number to be 10, and so on. Just be careful not to force the left or right middle boxes (8-10-10-10 and 5-21-5-19) to sum to T, and you will successfully complete the magic square.

The same process will work for Construction 1 of Figure 4. All you need to memorize is to start with $B - 1$ and $C - 2$, and the rest of the square can be finished in exactly the same order as the previous example. Even the general problem of three arbitrary squares can be handled by this approach. With three arbitrarily chosen squares and numbers, it is always possible for the magician to choose two more squares and fill them with arbitrary numbers (or chosen by audience members), and then the rest of the magic square is forced by exploiting sets of four squares as above.

For example, if a member of your audience (perhaps a Tau-ist) gave you the numbers 6, 28, and 31 in Figure 13(a), with prescribed total 85, you could place any number, say 30, in the square above 31, forcing the square above that to be 18. Now place your next arbitrary number, say 7, anywhere that will cause a chain reaction, and this will complete the rest of the square. For instance, by placing it between 6 and 28, you force the number 44 (top row), then 54 (upper left quadrant), −4 (top quadrant), 17 (second row), 14 (diagonal), and so on, resulting in Figure 13(b). (Again, avoid using the left and right quadrant, but you can use the middle quadrant or the four corners.)

(a)

6		**28**	
31			

(b)

6	7	**28**	44
18	54	−4	17
30	14	21	20
31	10	40	4

Figure 13. Another way to complete a magic square.

The "basis" for this method is that the vector space of 4×4 magic squares has dimension 7. (In [**6**], James Ward showed that the vector space of $n \times n$ magic squares has dimension $n^2 - 2n - 1$.) If we additionally impose the symmetry requirement of the upper left quadrant and the top quadrant (and their reflections), then that brings the dimension down to 5. Thus it is always possible, after three (necessarily independent) choices of squares, to find two other squares that will force the rest of the magic square. As a rule, you should avoid picking squares for your free choices that immediately put four chosen squares as vertices of a rectangle. Although this method requires no memorization, we prefer the original method since it requires less calculating effort, and can be created much faster with less potential for error.

We note that, as an alternative to Construction 2R, the magician can rotate Construction 2 in either direction 90 degrees, and every one of the 24 remaining cases can be handled by one of these rotations (16 of them without parity restrictions). For the purist who does not wish to restrict the parity of any of the entries, to cover those remaining eight situations, you can use Construction 3 in Figure 14 or one of its reflections.

We have found the creation of challenging magic squares to be very popular with audiences, especially those with some mathematical aptitude. On one occasion, the second author performed for a science convention, and asked for three numbers and locations from a biologist, an astronomer, and someone who liked science in general.

A	B	C	D
$C+3x$	$D-3x$	$A+3x$	$B-3x$
$D-2x$	$C-2x$	$B+2x$	$A+2x$
$B-x$	$A+5x$	$D-5x$	$C+x$

Figure 14. Construction 3.

He then proceeded to create a magic square with a secret total of 42. He explained that such an outcome was inevitable because the people who supplied the numbers represented life, the universe, and everything [**1**]!

Summary. We present several effective ways for a magician to create a 4-by-4 magic square where the total and some of the entries are prescribed by the audience.

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binomial The prefix *bi-* is a Latin abbreviation of the adverb *bis, twice*. It should therefore be prefixed only to Latin words or words of Latin origin. The Greek noun *νόμος* means *rule* or *law*. Some claim that the word is legitimate because the second component is from the Latin *nomen, name*, with the adjectival ending *-alis* added, but this is unlikely, for then how does one explain the absence of the second *n*? What happened was that the Latin adjectival ending *-alis* was illiterately appended to the Greek noun, and the word became legitimate through its adoption by Newton, from whose authority there is no appeal.

—Anthony Lo Bello,
Origins of Mathematical Words,
Johns Hopkins University Press, 2013,
to be reviewed in the May issue.