Mathematics and Resource Management

Colin W. Clark
The University of British Columbia

The world's economic system depends utterly on the supply of natural resources. This is an obvious fact, but what is not so obvious is whether the world's stock of resources can be exploited sustainably over the long run. Will fossil fuels (oil and gas) eventually run out, and if so, where will we then obtain our energy? Can agricultural production be increased without bound, or is there a maximum level that cannot be surpassed? Are environmental resources such as the atmosphere and the oceans currently under stress to an unsustainable degree?

Ideally, we would like quantitative answers to these questions, but this quest is beyond our present capability.

Here I will address a much simpler (but far from simple) question: how can we manage a given renewable resource, such as a marine fish population, or a forest, in a sustainable way? And what forces tend to prevent sustainable harvesting? Can we identify an "optimal" harvest strategy, in some sense, and how could it be implemented?

These questions have been, and probably must be addressed by using mathematical models. Here is a simple example which has sometimes been used in fisheries:

$$\frac{dx}{dt} = G(x) - h(t) \tag{1}$$

where t denotes time, $x = x(t)$ denotes the size (biomass) of the population at time t, $G(x)$ denotes the natural net growth rate of the population, and $h(t)$ denotes the rate of harvesting ($h(t) \geq 0$). The growth function $G(x)$ is often assumed to be parabolic:

$$G(x) = rx\left(1 - \frac{x}{K}\right)$$

This is called the "logistic" model of population dynamics (e.g. Brauer and Castillo-Chávez 2001).

Supposing that the growth function $G(x)$ has been estimated in some way, we can then calculate the "maximum sustainable yield" or MSY:

$$\text{MSY} = \max_x G(x) \tag{2}$$

If you're a math student you should be able to prove this for yourself, using Eq. (1).

If the maximum in Eq. (2) occurs at $x = x_0$, then harvesting at the constant rate $\text{MSY} = G(x_0)$ will maintain the stock $x(t)$ at that level in perpetuity. Or will it? What if $x(0) < x_0$? Such a fish population would be considered to have been overfished – a common situation in marine populations today.

So now we need a theory to explain, and hopefully prevent, overfishing. In 1954 the Canadian economist H.S. Gordon came up with such a theory (Gordon, 1954). Gordon used the standard equation

$$h(t) = qx(t)E(t) \tag{3}$$

where $E(t)$ denotes "fishing effort" at time t – for example, the total number of nets in the water at that moment, and q is a constant, called catchability. Gordon let p denote the sale price of fish, and c the unit cost of effort, so that the fishermen's combined net revenue flow at time t equals

$$R(t) = ph(t) - cE(t) = (pqx(t) - c)E(t) \tag{4}$$

Prediction: In the absence of any control over fishing, the fish population will be fished down to the level x_{BE} given by

$$x_{\text{BE}} = \frac{c}{pq} \tag{5}$$

Here BE stands for "Bionomic Equilibrium."

Why is this prediction valid? Simply put, whenever $x(t) > x_{\text{BE}}$ we have $R(t) > 0$ - fishing is profitable. Therefore $E(t)$ will tend to be large, so that $dx/dt < 0$. But once $x(t)$ falls to (or below) x_{BE}, we get $R(t) \le 0$, so fishing ceases. QED

Looking again at Eq. (5), note that x_{BE} is proportional to cost c, and inversely proportional to price p. Therefore lower costs, or higher prices, imply more intensive exploitation of the resource. This implies more intensive overfishing, once we have $x_{\text{BE}} < x_0$. This, if you will, is just the reverse of what one expects in a free economy – higher prices or lower costs should imply an increase in productive capacity x, not a decrease.

Why is natural resource economics "upside down" in this way? Gordon attributes this to the fact that the fishery (in his model) is assumed to be common property. No one owns the fish population, so no one has an incentive to protect it. The "tragedy of the commons" (Hardin 1968) spells doom for all.

Gordon's theory seems almost childishly simple. But overexploitation of renewable resources is widespread today. It's not just fish. Tropical forests, rare plants, birds, and even the environment itself, are under devastating threat. Human understanding of these complex bio-economic systems remains at a low level. For further discussion please see Clark (2010). The journal *Natural Resource Modeling,* produced by the Resource Modeling Association (www.resourcemodeling.org) publishes ongoing studies in this area.

Brauer, F. and Castillo-Chávez 2001. *Mathematical Models in Population Biology and Epidemiology.* Springer (New York, N.Y.).

Clark, C.W. 2010. *Mathematical Bioeconomics: the Mathematics of Conservation.* Wiley-Interscience (New York, N.Y.).

Gordon, H.S. 1954. The economic theory of a common property resource: the fishery. *Journal of Political Economy* **62**, 124-142.

Hardin, C. 1968. The tragedy of the commons. *Science* **162**, 1243-1247.