# Jump Shot Mathematics

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**Abstract**

In this paper we examine variations of standard calculus problems in the context of shooting a basketball jump shot. We believe that many students will find this more interesting than the use usual manner in which such problems are presented in textbooks.

**Angle of elevation 60 degrees**

Suppose a basketball player takes a 15 foot jump shot, releasing the ball from a height of 10 feet and an angle of elevation of 60 degrees. What is the initial speed $V_0$ needed for the shot to go in?

If we neglect air resistance, this is a typical ballistic motion problem. The equations are [1]

$$x(t) = V_0 \cos(\theta)\, t,\; y(t) = -\frac{gt^2}{2} + V_0 \sin(\theta)\, t + h_0.$$

For this problem we have $g$=32 ft/sec$^2$ and $\theta = 60^{\circ}$. Since the ball is released from the height of the basket, we can take $h_0$ to be zero.

The range is given by

$$d = \frac{{V_0}^2 \sin(2\theta)}{g}.$$

If we set $d$ = 15 feet and solve for $V_0$ we get,

$$V_0 = \sqrt{\frac{15*32}{\sin(120^{\circ})}} \cup 23.5426 \text{ ft/sec.}$$

Suppose that later in the game, the player takes another jump shot from the same position but, maybe because she has tired, takes the shot with an initial angle of elevation of 30 degrees. What is the initial velocity needed this time?

Since the angles are complementary, the initial speed will be the same:

$$V_0 = \sqrt{\frac{15*32}{\sin(60^{\circ})}} \cup 23.5426 \text{ ft/sec.}$$

Figure 1 shows the path of the two shots.

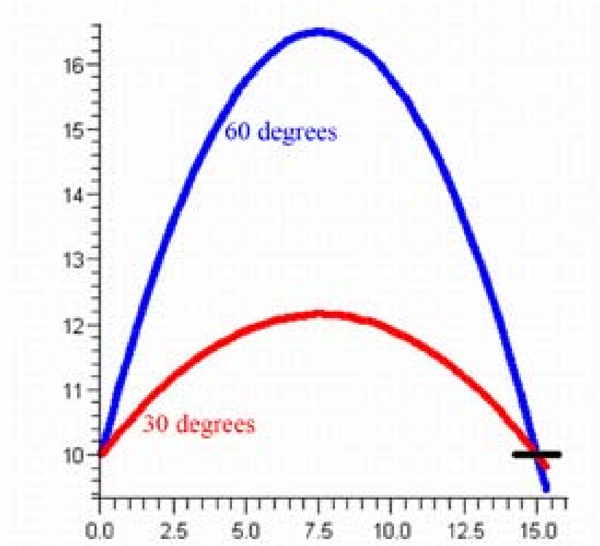

Figure 1

In Figure 1, the lower curve is closer to the front rim of the basket than the upper curve. Basketballs are not points. According the Wikianswers.com [2], the rim of a basket has a diameter of 18 inches. A woman's basketball has a diameter of approximately 9 inches and a men's basketball has a diameter of about 9.4 inches. For the rest of this paper, we will assume that the shooter is a woman basketball player.

We are interested in how close the center of the basketball comes to the front rim or back rim. Put another way, does the shot result in nothing but net?

This is a variation of an optimization problem seen in most calculus books [1], namely, finding the minimum distance from a point not on a curve to the curve. We believe that students do not find these problems interesting, but some students will be interested in the answer to our basketball problem. For the shot at an angle of 60 degrees, the equations of motion are

$$x(t) = V_0 \cos(60^\circ)\, t$$
$$y(t) = -16t^2 + V_0 \sin(60^\circ)\, t.$$

The distance from the front of the rim is given by the minimum of

$$Df = \sqrt{(x(t) - 14.25)^2 + (y(t))^2}.$$

For *Db*, the distance from the back of the rim, 14.25 is replaced by 15.75.

The results are $Df_{\min} = 7.77$ in and $Db_{\min} = 7.82$ in.

The shot easily goes through the basket touching nothing but net using either the man's or the woman's basketball.

**Angle of elevation 30 degrees**

With an initial angle of 30 degrees, the equations are the same except that 30 degrees replaces 60. We will leave it as an exercise to verify that $Df_{\min} = 4.37$ in and $Db_{\min} = 4.62$ in.

Since the radius of the ball exceeds 4.37 in, it hits the front of the rim, but would clear the back of the rim with room to spare. Figure 2 shows the paths near the basket with the balls and basket superimposed.

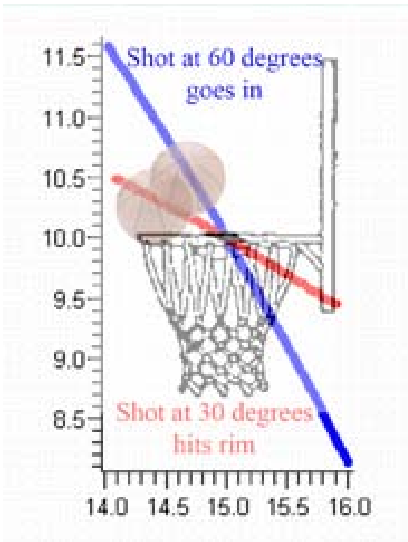


Figure 2

Perhaps we can make the shot if the center of the ball goes through a point a little closer to the back of the rim. After experimenting with changing the initial velocity, we found that with an initial speed is 23.56 ft/sec we obtain $Df_{\min}$ = 4.49 in and $Db_{\min}$ = 4.48 in. Since the radius of the basketball is 4.5 in, the shot grazes both the front and back rims. Thus it is not possible to get nothing but net shooting at an angle of elevation of 30 degrees from 15 feet if the shot is released from a height of 10 feet.

This raises the question: how close to 30 degrees can we get?

Solving the equations for a shot through the center of the basket with an initial angle of 31 degrees yields

$$V_0 = 23.31 \text{ ft/sec},$$
$$Df_{\min} = 4.505 \text{ in},$$
$$Db_{\min} = 4.756 \text{ in},$$

so we can conclude that the ball clears the rim and the shot is good.

**Varying the distance**

What if the shot is taken from a distance other than 15 feet? When we first considered this problem, it seemed that we would have to solve the equations with an additional variable, the distance from the basket. We can make the problem simpler if we imagine a defender trying to block the shot and having the tips of her fingers at a height of 10 feet and a distance of 9 inches from the center of the ball at release. Since the release point and basket are the same height, the minimum distance of the ball from the front of the rim is exactly the same as the minimum distance of the ball from our imaginary defender.

If we eliminate the parameter by solving for $t = x/(V_0\sqrt{3}/2)$ and substitute into $y(t)$, we get

$$y = \frac{-16x^2}{V_0{}^2} + \frac{\sqrt{3}\,x}{3}.$$

This is the equation of a parabola opening downward, so the path is concave down and lies below its tangent line at $x = 0$. As we increase the distance from the basket and hence the initial velocity, the path approaches that tangent line. The equation of the line tangent to the curve at $x = 0$ is

$$y = \frac{\sqrt{3}\,x}{3}$$

and the minimum distance from the defender's hand to that line is

$$Df_{\min} = 4.5 \text{ in}.$$

Trigonometry makes the minimum distance to the defender's hand obvious since it is the length of the side opposite a 30 degree angle in a right triangle, and half the length of the hypotenuse, the distance from the center of the ball at release to the defender's finger tips, 9 in. Since the actual path of the ball lies below this tangent line, the minimum distance to the defender's fingers must be less than 4.5 in and the shot is blocked.

Figure 3 shows the paths ($0 < x < 0.75$) of the center of basketballs shot with initial velocities of 5, 10, 20 ft/sec and the line tangents to them at $x = 0$. The tip of the defender's fingers is the lower right corner of the graph.

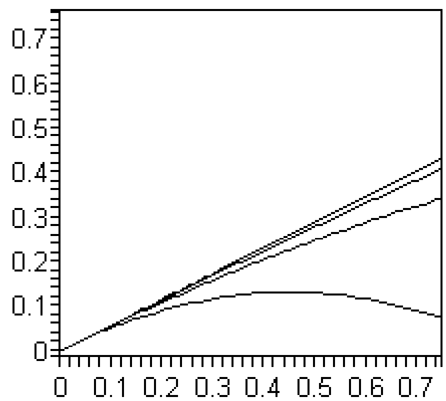


Figure 3

If we view shots taken at different initial velocities at an angle of 30 degrees as the basketball passes through the center of the basket, we get the mirror image of Figure 3. Thus any shot at an initial angle of elevation of 30 degrees that goes through the center of the basket will have a minimum distance from the front rim less than the 4.5 in radius of the ball and will therefore touch the front rim.

**Varying the height**

Another variation is to consider a shot from 15 feet at an initial height different than 10 feet. Since the path is a parabola, the closer we are to the vertex, the smaller the angle of elevation. If a shot is taken from a height of less than 10 ft, the angle at the basket will be less than 30 degrees and thus can't clear both rims. If the shot is taken from a height greater than 10 ft, the angle at the basket will be more than 30 degrees and it will be possible to miss both the front and back rims. For example, if the jump shot is from a height of 9 ft, the initial speed will be approximately 25.03 ft/sec, the angle of elevation at the basket is approximately -23.94 degrees, and $df_{\min} \cup 3.50$ in. On the other hand, if the shot is taken with an initial height of 11 ft, the initial speed will be approximately 22.29 ft/sec, the angle of elevation at the basket is approximately -35.40 degrees, and $df_{\min} \cup 5.10$ in. Since the basketball has a radius of 4.5 in, the first shot hits the rim while the second one clears the rim and scores.

What does this say about a shot at an initial angle of elevation of 30 degrees? First, we see that it is not possible to make the shot unless the initial height is more than 10 feet. Second, the shot at the low angle is more likely to be blocked and third, a higher angle of elevation means that the ball will be in the air longer so you and your team will have more time to get into position for the rebound.

After all this there *is* a way to make a shot from a height of 10 feet with an initial angle of elevation of 30 degrees. The ball must be shot with the center of the basketball at a distance from the center of the rim less than 4.5 inches! Of course, this requires the player to be under the basket and shot from inside it.

References

1. J. Stewart, Calculus, *Early Transcendentals 6th ed.,* Thompson, Brooks/Cole. Belmont, CA 2008

2. Wikianswers available at www.wikianswers.com

Howard Penn received his B.A. in Mathematics from Indiana University in 1968 and his Masters and PhD in Mathematics from the University of Michigan in 1969 and 1973. Since receiving his final degree, he has taught at the United States Naval Academy where he now holds the rank of Professor. He is one of the pioneers is the use of computer graphics in the teaching of mathematics, having written the program, MPP, which was widely used internationally. He is also interested in finding applications of the mathematics to areas that many students will find interesting. Outside of mathematics, he is well known amateur photographer with special interests in wildlife, landscapes and abstract photography. His work may be viewed at www.flickr.com/photos/howardpennphoto.