THE SCIENCE OF A DRIVE

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ABSTRACT. Golf provides numerous examples of common physical phenomena which can be elucidated through mathematics. This notes provides a simple introduction to mathematical modeling in golf, by briefly describing a few of the many ways mathematics can be used to understand or improve the golf drive. First we describe the double-pendulum model of a golf swing, which is a simple but useful model of the mechanical system consisting of the golfer and the golf club, used to accelerate the club head. Second we consider the basic mechanics of the energy and momentum transfer which takes place when the club head impacts the golf ball. Finally we describe the thre basic forces—gravity, drag, and lift—which determine the ball's trajectory after it is struck by the club.

"Math and science are everywhere." With those words, championship golfer Phil Mickelson began a public service television advertisement produced by ExxonMobil and premiered during the 2007 broadcast of the Masters Golf Tournament. I had the privilege to serve as the mathematical consultant for the ad and for the accompanying website, *The Science of a Drive*, from which the title of this article is taken, and which can still be viewed at www.exxonmobil.com/Corporate/Imports/scienceofadrive/. Figure 1 displays a still frame taken from the advertisement and another taken from the website.

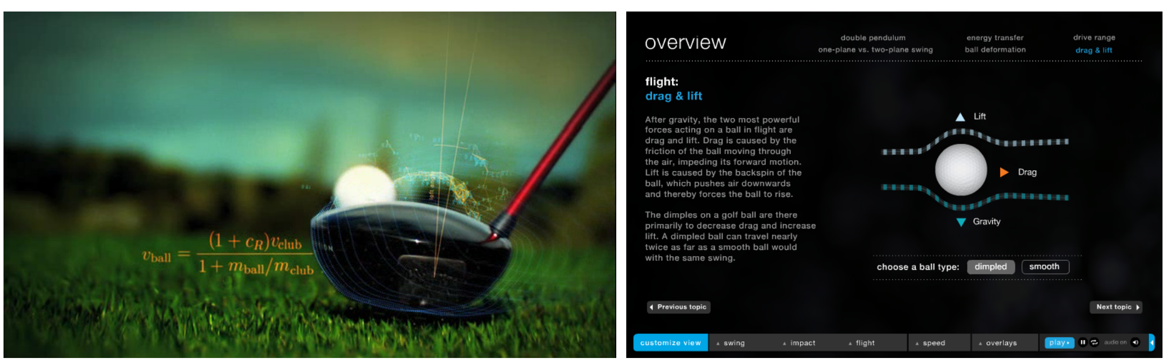

FIGURE 1. Frames from the television advertisement and the website.

The golf drive does indeed provide numerous examples of the ways mathematics elucidates common physical phenomena. Many aspects of it can be illuminated or improved through mathematical modeling and analysis of the mechanical processes entering into the game. Here I present a few simple examples collected during my consulting work. Specifically I briefly discuss three applications of mathematical modeling to fundamental mechanical processes in the golf drive: the double-pendulum model of a golf swing, transfer of energy and momentum in the club head/ball impact, and drag and lift in the flight of the golf ball.

These examples just scratch the surface of the subject. Indeed, there is a large literature on the subject of mathematics and mechanics of golf. See, for example, the survey [5] which discusses several aspects:

- models of the golf swing,
- the physics of the golf club and ball,
- the impact of the club head and the golf ball,
- golf ball aerodynamics,
- the run of the golf ball on turf.

1. The double-pendulum approximation of the swing

When a golfer swings for a long drive, the goal is to accelerate the club head so that it impacts the ball at just the right point, going in just the right direction, and moving as quickly as possible. To do so, the golfer exerts force with his or her arms on the shaft of the club, which in turn exerts force on the club head. This situation may be approximated as a double pendulum as depicted in Figure 2. The arms, pivoting at the shoulders, roughly behave as a pendulum, and the hands, grip, and shaft, pivoting at the wrists, behave as a second pendulum attached at the end of the first. For a well-timed drive, at the moment of impact the upper pendulum—the arms—is swinging very rapidly about its pivot point, and, at the same moment, the club is swinging very rapidly around its pivot point. These movements combine to accelerate the club head to speeds as high as 120 miles per hour.

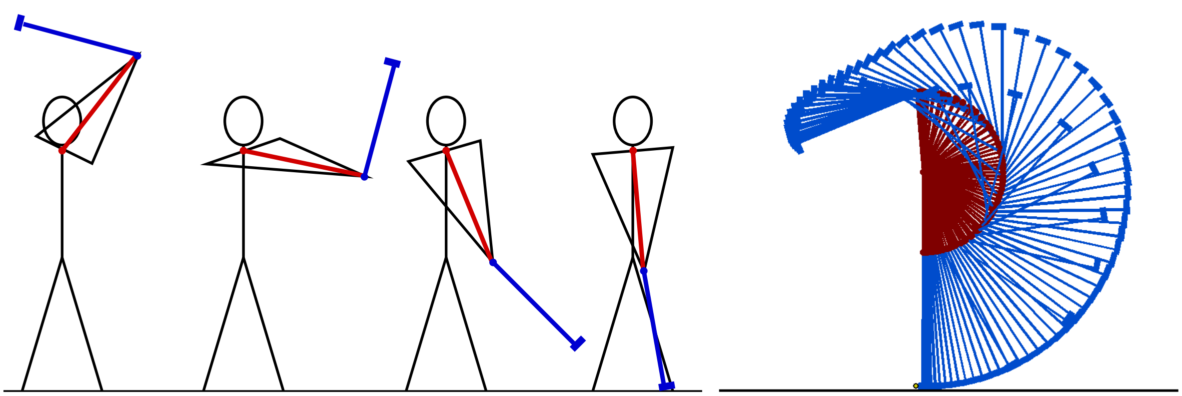

Figure 2. The double pendulum model of a golf swing.

Of course the double pendulum model is a crude approximation of the complex mechanism formed by the body and the club during a swing. The model can be refined in many ways, for example by taking into account the movement of the shoulders (and so of the pivot point of the upper pendulum) [7], the flexing of the club shaft [3], and the three-dimensional aspects of the motion [4].

2. The impact of the club head and the ball

The velocity of the club head, together with its mass, determine its kinetic energy and momentum. As the swing progresses, the golfer applies more and more force to the club head causing it to accelerate and so increase its speed. Therefore its momentum and energy increase. Upon impact, some of this energy and momentum is transferred to the ball. To determine the speed of the ball as it leaves the tee, we use conservation of both energy and momentum. Let m_{club} and m_{ball} denote the mass of the club and the ball, respectively. Let V_{club} and v_{ball} denote their speeds right after impact, and let v_{club} denote the speed of the club head just before impact. (Of course the speed of the ball just before impact is zero.) Since $E = mv^2/2$, conservation of energy tells us that

$$\frac{1}{2}m_{\text{club}}v_{\text{club}}^2 = \frac{1}{2}m_{\text{club}}V_{\text{club}}^2 + \frac{1}{2}m_{\text{ball}}v_{\text{ball}}^2,$$

while conservation of momentum tells us that

$$m_{\text{club}}v_{\text{club}} = m_{\text{club}}V_{\text{club}} + m_{\text{ball}}v_{\text{ball}}.$$

The solution to these equations is easily found:

$$V_{\text{club}} = v_{\text{club}}\frac{m_{\text{club}} - m_{\text{ball}}}{m_{\text{club}} + m_{\text{ball}}}, \quad v_{\text{ball}} = v_{\text{club}}\frac{2m_{\text{club}}}{m_{\text{club}} + m_{\text{ball}}} = v_{\text{club}}\frac{2}{1 + m_{\text{ball}}/m_{\text{club}}}.$$

Thus the ratio of the ball speed to the speed of the club head before impact is $2/(1 + r)$ where r is the ratio of the mass of the ball to the the mass of the club head. Notice that, no matter how small the ratio of masses, the ball speed will always be less than twice the club head speed. For instance, if $v_{\text{club}} = 54.0$ meters per second (about 120 miles per hour), $m_{\text{club}} = 0.195$ kilograms, and $m_{\text{ball}} = 0.0459$ kilograms, then v_{ball} is about 87.4 meters per second or just about 195 miles per hour.

In reality, not all of the kinetic energy lost by the club head during impact is converted into kinetic energy of the ball. That is, the impact is not perfectly elastic. Some energy is lost to heat and damage to the ball. In this case, the ball launch speed is given by

$$v_{\text{ball}} = \frac{(1 + c_R)v_{\text{club}}}{1 + m_{\text{ball}}/m_{\text{club}}} \tag{1}$$

where c_R is called the *coefficient of restitution*. For an elastic collision, $c_R = 1$, but in reality it is somewhat smaller. Using a typical value of $c_R = 0.78$, we obtain a launch velocity $v_{\text{ball}} = 77.8$ meters per second, or about 175 miles per hour. Even to the nonspecialist, formula (1) conveys a sense that math impinges on golf, and it was prominently displayed in the television advertisement (see Figure 1).

The period of contact of the club head with the ball is about one two-thousandth of a second. During this time the center of mass of the ball has barely moved, but the ball is bent way out of shape. A significant portion of the kinetic energy has been converted into potential energy stored in the deformed ball. Essentially, the ball is like a compressed spring. See Figure 3. When the ball takes off from the tee, it returns to a spherical shape, releasing the spring, and most of this potential energy is converted back into kinetic energy. Detailed analyses of the club head/ball interaction can be made through a full 3-dimensional finite element analysis [2] or via simplified 1- or 2-dimensional models [1].

3. The ball's flight

Once the ball is in flight, its trajectory is completely determined by its launch velocity and launch angle and the forces acting on it. The most important of these forces is, of course, the force of gravity, which is accelerating the ball back down towards the ground at

FIGURE 3. Golf ball under compression from impact of club on left.

9.8 meters per second per second. But the forces exerted on the ball by the air it is passing through are important as well. To clarify this, we choose a coordinate system with one axis aligned with the direction of flight of the ball and the others perpendicular to it. Then the forces exerted by the atmosphere on the ball are decomposed into the *drag*, which is a force impeding the ball in its forward motion, and the *lift* which helps the ball fight gravity, and stay aloft longer (Figure 4). Drag is the same force you feel pushing on your arm if you stick it out of the window of a moving car. Lift is a consequence of the back spin of the ball, which speeds the air passing over the top of the ball and slows the air passing under it. By Bernoulli's principle the result is lower pressure above and therefore an upward force on the ball.

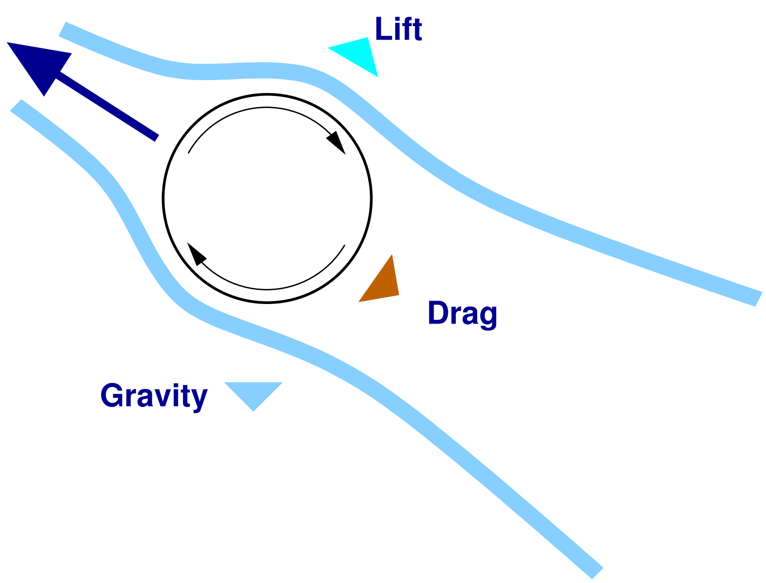

FIGURE 4. Forces acting on a golf ball during its flight.

Drag and lift are very much affected by how the air interacts with the surface of the ball. The dimples on a golf ball are there primarily to decrease drag and increase lift. Proper dimpling of a golf ball induces turbulence in the boundary layer, delaying the point at which the flow past the ball separates from the surface, and resulting in a ball which can carry nearly twice as far as a smooth ball would with the same swing. Based on aerodynamic and manufacturing considerations, a great many dimple designs have been manufactured, leading to an elaborate crystallography of golf balls [6].

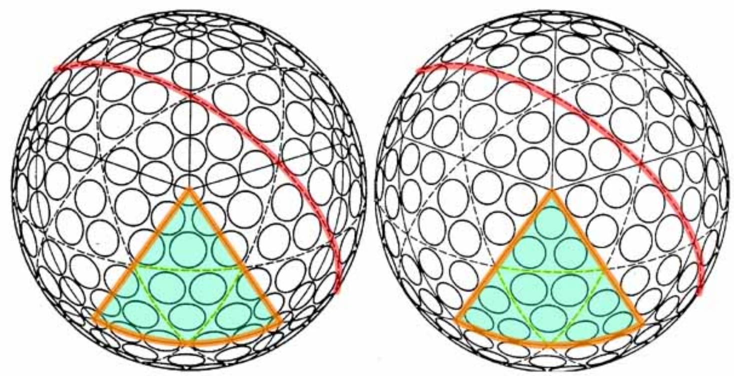

FIGURE 5. Two golf ball dimple patterns with icosahedral symmetry.

Mathematics Awareness Month 2010, with the theme Mathematics and Sports, provides us mathematicians with another opportunity to get out the vitally important message that mathematics can be found everywhere in the physical world and human activity. In this note, I have discussed briefly a few of the ways in which mathematics relates to golf. All of these could be, and in fact have been, the subject of extended studies, because they enable us not only to better understand, but also to optimize, the performance of a golfer and golf equipment.

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