Mathematics and Climate: The computer models used by scientists to forecast the weather and climate rely heavily on mathematics to accurately represent the various physical processes governing the motion of the Earth's atmosphere. In this brief essay, we describe a simple physical experiment which is being used to improve our mathematical methods for prediction.

Chaotic Convection: Much of the weather we experience is the result of natural *convection*, whereby solar energy is re-radiated from the Earth's surface to the nearby air, causing it to rise. As the warm air rises it begins to cool, its water vapor condenses into clouds, and the cool air eventually sinks. In the early 1960's, MIT meteorologist Edward Lorenz was looking for a simple mathematical model of this process of convection. He was hoping to demonstrate to the scientists at the National Weather Service that their methods of linear prediction were inadequate for the problem of short-term weather prediction. He settled on a system of three Ordinary Differential Equations (ODEs) and described numerical solutions to the system in a groundbreaking paper [1]. The solutions exhibit sensitive dependence on their initial position, leading virtually indistinguishable states to diverge quickly on small spatial scales. This phenomenon, which became known as *chaos*, is a major contributor to inaccuracies in weather and climate forecasts.

The Experiment: A thermal convection loop is a simple model of convection in the Earth's atmosphere in the form of a donut shaped tube, filled with fluid, and oriented vertically like a wheel. The bottom (top) half of the tube is warmed (cooled) uniformly by a bath of hot (cold) water; the forcing does not change with time. Once buoyancy overcomes viscosity, the geometry of the thermal convection loop forces the fluid to rotate rather than to simply rise and mix. Initially stationary, the fluid will accelerate until the effects of buoyancy, gravity, and friction balance each other out and the fluid reaches a steady state. In this state, fluid in the tube rotates in only one direction from the moment conduction gives way to convection. However, under certain conditions, a steady state is never reached by the fluid. When the constant temperature difference between the heat sink along the top half and heat source at the bottom half is within a particular range, the fluid oscillates in a chaotic manner. See Figure 1 for an illustration of the experimental apparatus [2].

Positive Feedback: To visualize the instability, assume the fluid is in an equilibrium state of counterclockwise flow and that an anomalous *hot* pocket of fluid arrives at point B in the tube. The hot pocket exerts a buoyancy force on the fluid, causing a positive acceleration in the counterclockwise direction and speeding up the rotation (arrow a). This pocket cools less on its journey across the top half of the tube, arriving at point A hotter than it should be (due to the thermal anomaly). Consequently, there is a buoyancy force in the reverse (clockwise) direction acting to slow down the speed of rotation (b). The fluid decelerates, but continues in the counterclockwise direction, allowing the anomalous pocket more time to heat up. Upon arriving at point B, the pocket is once again hotter than it was previously, by a greater amount now than before. The buoyancy force acting to accelerate the flow is greater this time, since the instability has been magnified (c). With even less time to cool off as the pocket crosses the top half of the tube, the instability continues to grow, decelerating the clockwise flow again (d). Amplification continues until the buoyancy deceleration force generated by the pocket at point A grows large enough, causing the flow to stop. With no rotation in the tube, the temperature gradient between the top and bottom hemispheres of the fluid grows undisturbed. The fluid will then 'choose' a direction of rotation and repeat the behavior indefinitely. When differential heating is below the critical value, thermal anomalies decay in time.

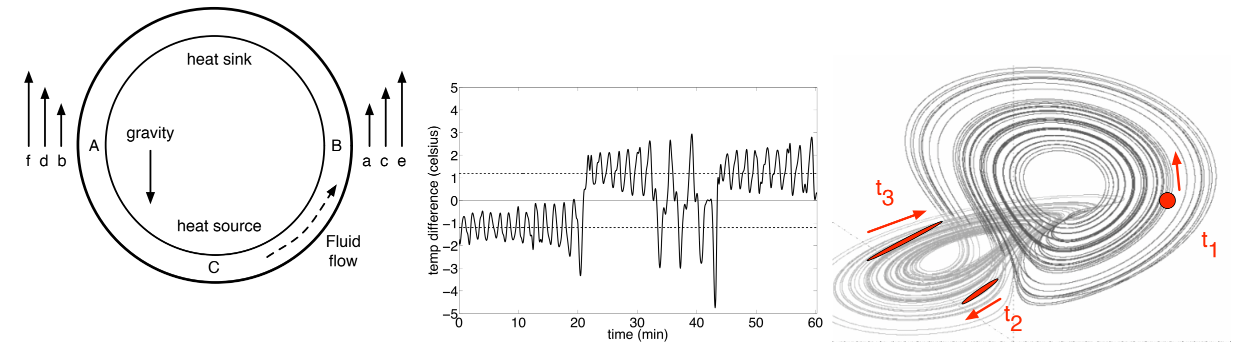

Figure 1: (Left) Cartoon illustrating the experimental apparatus, a thermal convection loop. Uniform heating across the bottom half of the loop generates steady convection, which gives way to chaotic flow once a threshold temperature difference is reached. (Middle) Measurements of the temperature difference between points A and B as a function of time [3]. (Right) The Lorenz Attractor, the typical shape of solutions to Lorenz's 1963 model for convection. The dynamics governing small ensembles of initial conditions in the toy climate will be used to computationally predict flow reversals in the experiment.

Applications: The Fourth Assessment Report of the Intergovernmental Panel on Climate Change (IPCC FAR) recently concluded that "most of the observed increase in globally averaged temperatures since the mid-20th century is very likely due to an increase in anthropogenic greenhouse gas concentrations." Forecasting the climate response to anthropogenic perturbations (e.g. increased CO_2) requires characterization of a complex system of feedbacks, many of which are not well understood. Nevertheless, scientists hope to predict the average global temperature rise, among other quantities, throughout the next century and beyond. As the climate modeling community explores future scenarios of global warming, uncertainty estimates generated by computer simulations *must* be more efficient and accurate. Simple systems like the thermal convection loop are being used as a testbed on which to develop and refine these mathematical techniques.

The Model: The equations of Lorenz (1963) are given by $\dot{x} = -\sigma(x-y)$, $\dot{y} = -xz - rx - y$, $\dot{z} = xy - bz$, where x is proportional to the flow velocity, y is proportional to the temperature difference between points A and B, and z is proportional to the deviation of the convecting temperature profile from that of conduction (linear). The parameter b is related to the geometry of the tube, σ is the Prandtl number (ratio of viscosity to thermal diffusivity), and r is the Rayleigh number (critical value distinguishing conduction from stable convection and chaotic convection).

Links: For more information visit the *weather chaos page* at the University of Maryland or the *research* page of C. M. Danforth where you can download MATLAB code to make forecasts with Lorenz's model.

References:

[1] E. Lorenz. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20:130–141, 1963.

[2] H. F. Creveling, J. F. De Paz, J. Y. Baladi, and R. J. Schoenhals. Stability characteristics of a single-phase free convection loop. *Journal of Fluid Mechanics*, 67:65–84, 1975.

[3] C. M. Danforth. Why the weather is unpredictable: An expermental and theoretical study of the lorenz equations. *Honors Thesis in Mathematics and Physics, Bates College*, 2001.