Mathematics and Voting*

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The 2008 annual *Math Awareness Month* theme features the unusual combination of "Mathematics and Voting." The importance of voting is obvious; indeed, with the United States election season hard upon us, discussions about voting are seemingly nonstop. More generally, on almost any day of any year the news media reports on some consequential election going on somewhere in the world. But what does mathematics have to do with any of this?

Actually, a lot. In fact, mathematics has been central to this area since 1770 when the mathematician Jean Charles de Borda challenged whether the French Academy's elections for new members accurately reflected the views of the voters: Borda blamed their voting rule, the plurality vote. It is this concern, whether an election rule can faithfully produce outcomes capturing "the will of the voters," that requires serious mathematical attention. Consider the consequences; as it is well understood, electing the "wrong leader" can cause serious, lasting problems. But while the mathematical study of this topic is more than a couple of centuries old, there remain more mysteries than answers and the mathematical development of this area is in its early stages.

To illustrate what can happen, suppose in an election for the next chair of your department the voter preferences among candidates $\{A, B, C\}$ are

Number	**Ranking**	**Number**	**Ranking**
3	$A \succ B \succ C$	2	$B \succ C \succ A$
2	$A \succ C \succ B$	4	$C \succ B \succ A$

If your department uses the plurality vote-for-one rule, A wins with the $A \succ C \succ B$ ranking. But with a vote-for-two rule, the ranking *reverses* to become $B \succ C \succ A$ so formerly last-place B now wins. By using the *Borda Count* (introduced in 1770 by Borda) where a ballot is tallied by assigning a first and second positioned candidate, respectively, two and one points, C wins with the $C \succ B \succ A$ ranking. As *any* candidate can win with an "appropriate" voting rule, all three outcomes cannot register the will of the voters. Who should be the winner?

Accompanying the disturbing reality that *election outcomes can more accurately reflect the choice of an election rule than the voters' wishes* are the mathematical issues of understanding why this is so, determining whether "paradoxical outcomes" are unlikely anomalies or reasonably prevalent behavior, developing appropriate mathematical structures to permit a systematic rather than an ad hoc analysis, and identifying whether any election rule reliably produces outcomes that, arguably, represent the views of the voters. Can the reader, for instance, determine what mathematical structures force the radically different conclusions for the above example?

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In this article, I answer some of these questions while indicating mathematical structures currently being developed to analyze voting systems. More generally, we should treat the "mathematics of voting" as a prototype to identify mathematical concerns that are associated with aggregation rules. For instance, as I will indicate why orbits of symmetry groups and even notions from chaotic dynamics play roles in understanding voting problems, expect similar mathematical approaches to identify which non-parametric statistical procedures best represent the data and whether Adam Smith's Invisible Hand story from economics truly captures the wishes of consumers.

Before seeking solutions, we need to appreciate what kinds of problems can arise. Thus the first part of this article illustrates certain ways in which election outcomes can widely differ. After understanding what can go wrong, I will indicate a way to analyze these issues.

1 How bad can it get?

With the above 11 voter example, each candidate wins with some *positional voting rule*: this is a voting rule where an n-candidate ballot is tallied by assigning a specified weight w_j to the candidate in the j^{th} *position*, $j = 1, \ldots, n$. (The obvious constraints are that $w_j \geq w_{j+1}$ for all j and the weights are not all equal.) The choice matters; e.g., by using different weights, this introductory example has *seven* different election rankings; four of them are strict (without ties).

While it has been known since Borda that the choice of a voting rule matters, in order to approach this area systematically, we need results demonstrating whether the problem is serious. To do so, treat the weights as components of a *voting vector* $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ where $w_n = 0$. A normalized version (where $w_1 = 1$) of any voting vector is a convex combination of the "vote-for-k" vectors $(1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1, 0)$. Using this convexity with the geometry of $\mathbb{R}^n$ and the linearity of the tallying process, it is not overly difficult to prove the following disturbing conclusion.

Theorem 1 *(Saari [6]) For $n \geq 2$ candidates and any k satisfying $1 \leq k \leq (n-1)(n-1)!$, there is a profile (i.e., a list of each voter's complete, transitive ranking of the candidates) where, with different choices of the positional voting rule, precisely k different, strict election rankings arise. (No profile generates more than $(n-1)(n-1)!$ strict rankings.) Indeed, for $n \geq 4$, a profile can be found where each candidate is ranked in first, second, ..., last place with different positional rules.*

With three candidates, then, a profile can generate four different strict election rankings. But with $n = 10$, which is about the number of candidates starting in an US presidential election, a single profile could have over three million different election rankings where each candidate could be the "winner" with some rules, and the "loser" with others. The 2002 French presidential election started with 16 candidates; here a single profile could admit well over 19 *trillion* different positional election rankings. Somewhat surprisingly but because of the astronomical numbers that can arise with combinatorics, there are reasons to believe [10] that examples exhibiting trillions of outcomes can be created with, say, no more than 40 or 50 voters. The point to be made for this article is that if changing rules generates trillions of different rankings, the mathematical challenge is to determine which positional voting rules, if any, can be trusted to get the job done accurately.

As the "mathematics of voting" is a prototype, expect the Thm. 1 behavior to accompany other aggregation rules. As an illustration, if political parties A and B have, respectively, 49 and 51 of the senate seats and 217 and 218 of the congressional seats, one might mistakenly accept that they have essentially equal power. But as B enjoys a majority in both houses, it has complete power; i.e, simple counts are misleading; a more accurate measure is a group's ability to affect change. "Power indices" were developed in game theory to measure these differences in the ability of competing groups, or even individuals, to determine outcomes. These tools have been cited in Supreme Court decisions and used to analyze EU voting rules. While the best known ones are the Shapley and Banzhaf values, many others have been introduced. Because of their importance and alerted by what can happen with voting rules, one should worry whether the choice of an index matters. It does; by using Thm. 1, Laruelle and Merlin [2] extended its conclusion to power indices; i.e., with the same data or game theoretic structure, it is possible to have up to $(n-1)(n-1)!$ different rankings of the n parties by using different power indices. Independently and using different techniques, Saari and Sieberg [11] proved similar results.

What about that nagging concern whether these voting examples are cooked up anomalies or reflect what must be expected. For decades, researchers, such as Fishburn, Gehrlein, Riker and others have estimated the likelihoods of various voting paradoxes with consistently discouraging conclusions. These results were hampered by the complexity of the associated computations, which forced using unrealistic assumptions such as that each profile is equally likely. The next result, which is based on a six-dimensional central limit theorem and where the computational difficulties are handled by borrowing ideas from Schläfli [13], provides more realistic estimates for the issues described here while underscoring the severity of these problems. Incidentally, the reason a zero probability is associated with even k values is that, here, one of the tie rankings is easily broken.

Theorem 2 *(Saari and Tataru [12]) For three candidates, assume there are n voters where, as $n \to \infty$ the distribution of voter choices is asymptotically independent with an asymptotic common finite variance and the asymptotic mean has an equal distribution of voters of each type. The limiting probability as $n \to \infty$ that a profile permits precisely k different outcomes as the positional rule choice varies is zero if k is an even integer and*

k	***Probability***	***k***	***Probability***
1	0 : 31	3	0 : 44
5	0 : 19	7	0 : 06

With a surprisingly high probability of 0.69, then, the choice of the positional voting rule matters for close three-candidate elections! Different rules can have different outcomes. Indeed, it is easy to find many actual elections where the outcomes appear to reflect the voting rule rather than the voters' intent.

The mathematical approach Tataru and I developed for this theorem has been applied by others, including various combinations of Merlin, Tataru, Valognes, Gehrlein, and Lepelley, to obtain related three-candidate conclusions. While a definitive answer for four or more candidates has yet to be investigated, it is clear that the severity of the problem escalates with the number of candidates. For instance, with more than five candidates in a closely contested election, there are reasons to believe that, with probability close to certainty, the election ranking changes with positional methods.

2 Chaotic effects

To illustrate what else can go wrong, suppose during hiring season your department will make an offer to one of the four candidates $\{A, B, C, D\}$. Preferences within the department are

Number	**Ranking**	**Number**	**Ranking**
3	$A \succ C \succ D \succ B$	2	$C \succ B \succ D \succ A$
6	$A \succ D \succ C \succ B$	5	$C \succ D \succ B \succ A$
3	$B \succ C \succ D \succ A$	2	$D \succ B \succ C \succ A$
5	$B \succ D \succ C \succ A$	4	$D \succ C \succ B \succ A$

(1)

which define the plurality ranking of $A \succ B \succ C \succ D$. Just before contacting A to offer her a position, C calls to drop out because she just accepted a position elsewhere.

With C dropping out, should the decision be re-evaluated? With C's low ranking, I doubt whether any group would bother to hold another election. They should; two of C's previous voters would now vote for B and five for D creating the conflicting $D \succ B \succ A$ outcome! Indeed, with this example, if *any* one or any two of the candidates drop out, the plurality ranking reverses; it is compatible with the opposite $D \succ C \succ B \succ A$. So, is A or D the "true choice of these voters?"

The mathematical issue prompted by this example is clear. How bad can it get? What combinations of rankings for the different subsets of candidates can be actual election rankings? Combinatoric complexities discourage trying to explore this issue by creating examples. But as my research includes analyzing the dynamics of Newtonian N-body systems, a natural way to address this potentially chaotic state of affairs is to borrow notions from "chaos."

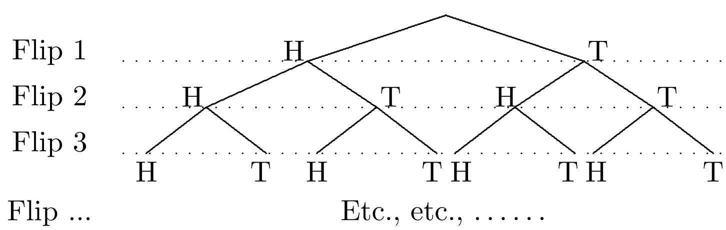


Figure 1. Randomness of flipping a coin

Start with a highly random situation; flipping a penny. The first flip outcome is either Heads or Tails. In either case, the second flip could be H or T. A tree, such as Fig. 1, indicates all possibilities: the mechanism of random behavior ensures that any branch of this tree, any listing of H's and T's, can arise. A sense of "chaotic dynamics" is created for a deterministic system if its orbits mimic the random behavior depicted by this tree by permitting all of the same possibilities. This can happen; e.g., a way to describe the orbits of the iterative system

$$x_{n+1} = f(x_n),$$

where the graph of f on the unit interval is given in Fig. 2, is with this tree of random coin-flipping behavior. On the tree, replace H with L for "Left," and T with R for "Right." For any sequence of L's and R's, there exists an initial point so that for each k, the k^{th} iterate is in the indicated region. Because each tree branch represents an admissible orbit, this deterministic system captures the randomness of Fig. 1. A power of this approach is that actual initial points for the deterministic system are not found; only their existence is verified.

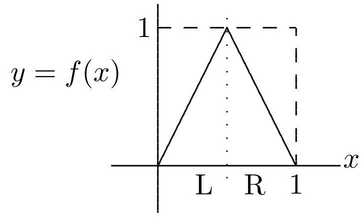


Figure 2. Randomness of a simple dynamic

To explore how to use a similar approach to determine whether a "chaotic state of affaris" occurs with voting rules, create a "tree of possible election outcomes." With three candidates $\{A, B, C\}$, start with the pair $\{A, B\}$ and draw three legs representing the three possible $A \succ B, A \sim B, B \succ A$ rankings. Pair $\{B, C\}$ has a similar three legged tree; append a copy to each of the three possible $\{A, B\}$ legs, and then do the same for $\{A, C\}$. This leads to a tree with 27 branches; what remains are the 13 rankings for the triplet $\{A, B, C\}$. Attaching this 13 legged object to each of the 27 branches creates a tree with 351 legs that lists all possible combinations of rankings over all subsets; denote this three-candidate tree by $\mathcal{T}^3$. Similarly, $\mathcal{T}^4$ represents the four-candidate tree that has over 1.4 billion legs. Just as with deterministic chaotic dynamics, a way to determine whether a sense of randomness can be associated with voting outcomes is to determine which branches, which combinations of rankings over the different subsets of candidates, can be realized as election outcomes with a profile.

Theorem 3 *(Saari [7, 8]) For any three-candidate positional voting rule where $w_1 \neq 2w_2$, anything is possible; for any branch on the $\mathcal{T}^3$ tree, there is a profile where the sincere election outcome of each subset is as specified. Only the Borda Count, where $w_1 = 2w_2$, does not admit all $\mathcal{T}^3$ branches; e.g., any branch on this tree where a candidate is ranked last in all pairwise rankings and ranked first in the ranking of all three cannot be a Borda outcome.*

As true with chaotic deterministic dynamics, but with the sole exception of the Borda Count, Thm. 3 indicates the "chaotic" sense allowed by election outcomes in that they permit anything to happen; e.g., a non-Borda positional election winner could even be the loser in "head-to-head" majority vote comparison with all other candidates. (Below I show how to create such examples.) Notice that by knowing which $\mathcal{T}^3$ branches are non-admissible identifies properties for the positional voting rule. For instance, some of the branches that are not permitted by Borda outcomes prove that it is the only positional rule where its outcomes are related to how the same voters majority rank the candidates in pairs.

With $n \geq 3$ candidates, we need terminology to represent which voting rules are used to tally elections with which subsets of candidates. Let $\mathbf{W}^n$ list the voting vectors assigned to tally the different subsets of candidates; after removing obvious redundancies, these vectors include an open set in an appropriate Euclidean space. So, if the plurality vote always is used, $\mathbf{W}^n$ is a concatenated list of plurality vectors. Let $\mathbf{B}^n$ represent where all elections are tallied with the Borda Count; i.e., the voting vector for each subset of candidates is such that the differences between successive weights is the same. Adopting terminology from dynamics, a list of rankings created by a profile, where the election for each subset is tallied as required by $\mathbf{W}^n$, is called a $\mathbf{W}^n$ *word.* For instance, the Eq. 1 plurality word consists of all ranking described above. The $\mathbf{W}^n$ *dictionary*, denoted by $\mathcal{D}^n(\mathbf{W}^n) \subset \mathcal{T}^n$, is the set of all possible $\mathbf{W}^n$ words. We now can state what happens in general.

Theorem 4 *(Saari [7]) For $n \geq 3$, there is a proper, lower dimensional algebraic variety $\mathcal{V}^n$ such* *that if* $\mathbf{W}^n \notin \mathcal{V}^n$*, then*

$$\mathcal{D}(\mathbf{W}^n) = \mathcal{T}^n. \tag{2}$$

However, $\mathbf{B}^n \in \mathcal{V}^n$*, and for all* $\mathbf{W}^n$ *where at least one voting vector is not a Borda vector,*

$$\mathcal{D}(\mathbf{B}^n) \subsetneq \mathcal{D}(\mathbf{W}^n). \tag{3}$$

Oh my; this result, which proves that the Eq. 1 example is exceedingly tame compared to what else can happen, explains my lost faith in the plurality vote. This is because all of the "vote-for-k" rules normally used in departmental and societal elections, including the plurality vote, are *not* in the algebraic variety $\mathcal{V}^n$. Consequently (from Eq. 2), any perverse listing of election rankings, even if the ranking for each of the $2^n - (n+1)$ different subsets of candidates is selected in a random manner, can actually occur! The good news is that the rare $\mathbf{W}^n$ systems belonging to the algebraic variety $\mathcal{V}^n$ are spared certain paradoxes combining weird conflicting rankings for different subsets of candidates; the Borda Count always belongs to $\mathcal{V}^n$. In fact, while the Borda Count admits some inconsistent behavior (I indicate how to find all examples), Eq. 3 proves that any list of questionable Borda election outcomes must also occur with any other collection of voting vectors.

At this point, please forgive me for tossing in a "gee-whiz" comparison to demonstrate the benefit derived by using the Borda Count over, say, the plurality vote. The plurality dictionary for seven candidates is identified with $\mathcal{T}^7$, so we could compare the cardinality of $|\mathcal{T}^7|$ with $|\mathcal{D}(\mathbf{B}^7)|$. While the difference $|\mathcal{T}^7| - |\mathcal{D}(\mathbf{B}^7)| > 10^6$ sounds impressive as it asserts that the plurality vote admits a million more paradoxical settings, a more impressive comparison is that $|\mathcal{T}^7|/|\mathcal{D}(\mathbf{B}^7)|$ exceeds a billion times the number of droplets of water in all oceans in the world. Thus Borda provides a shockingly higher level of consistency; the choice of a voting system matters! Incidentally, I have characterized the algebraic variety $\mathcal{V}^n$; e.g., in terms of voting outcomes, different $\mathcal{V}^n$ branches categorize those $\mathbf{W}^n$ that allow similar, new kinds of election inconsistencies.

Before explaining why these different election outcomes arise, it is worth indicating how these Thm. 4 results extend to other disciplines. In her thesis and a JASA article [1], Deanna Haunsperger used Thm. 4 to prove a similar result for non-parametric statistics. Namely, she replaced profiles with data sets, and positional rules with non-parametric statistical rules. As above, a "statistical dictionary" collects all lists of rankings, over different subsets of alternatives, that come from some data set by using a specified collection of statistical procedures. In proving that Thm. 4 extends to non-parametric statistics, Haunsperger showed that the Kruskal-Wallis rule assumes the role of the Borda Count. Thus, she proved that while the Kruskal-Wallis rule admits all sorts of new kinds of paradoxical behavior, when judged against all other choices, it is by far the most consistent!

The "algebraic variety" conclusion accurately suggests that the analysis involved different symmetry groups. Just by changing the groups, Thm. 4 extends to areas such as probability where the algebraic variety corresponds to various independence conditions. An interesting extension, where an algebraic variety kind of result has yet to be identified, is the Adam Smith "Invisible Hand" story. To introduce the ideas for a "pure exchange" model, prices for n commodities can be normalized to the "price sphere," which is the positive orthant of S^{n-1}. The aggregate excess demand function, which is the difference between the total demand and supply of each commodity at given prices, defines a continuous tangent vector field. Smith's story requires the vector field to have an attracting equilibrium. Contradicting this story was a stunning example created by H. Scarf where, instead, the only price equilibrium is a repeller forcing prices away from the equilibrium! This was followed by a Thm. 1 type result first recognized by H. Sonnenshein, advanced by R. Mantel, and posed in its current form by G. Debreu showing that beyond Scarf's example, "anything can happen." First, bound all prices away from zero by any specified $\epsilon > 0$. They showed it is possible to select *any* continuous tangent vector field to this price sphere, and then an initial endowment of goods and a nice utility function for each consumer can be found so that, with the ϵ restriction, the aggregate excess demand function agrees with the selected vector field.

For a result parallel to Thm. 4, I extended this conclusion to all possible subsets of two or more commodities. Namely, with n commodities, $a \geq n$ agents, and any $\epsilon > 0$, select any continuous tangent vector field for the appropriate price sphere for each subset of two or more commodities. There exists a utility function and an initial endowment for each agent so that, with each subset of commodities, the aggregate excess demand function agrees with the selected vector field within the ϵ restriction. Thus, with the same economic agents, the economics associated with different subsets of commodities need not have anything to do with each other; chaos reigns. From a mathematical perspective, the proof of this theorem involved creating appropriate continuous foliations, with appropriate convexity properties, to represent the level sets of the individual utility functions. My main point is that we must expect Thm. 4 type of results with aggregations.

3 Symmetry structures

The next step is to develop appropriate mathematical tools to analyze and explain all of these voting problems. The complexity is a dimensional issue; the $n!$ dimension of the domain, which is the space of preferences for n alternatives, quickly exceeds the dimension of the range, which is the space of election outcomes. As we know, a larger domain allows more kinds of outcomes: they become the voting paradoxes. So, a way toward creating a systematic analysis of voting issues is to develop appropriate structures to better understand the domain

Clues about what should be done are abundant throughout mathematics; e.g., in Galois theory, we are interested in elements that are fixed by automorphisms from particular subgroups. The approach adopted here is to find appropriate configurations of profiles that force a neutral, tied outcome for certain classes of voting rules, but non-tied outcomes for others. As an illustration, the orbit of the Klein four-group leads to the configuration

$$A \succ B \succ C \succ D, \quad D \succ B \succ C \succ A, \quad B \succ A \succ D \succ C, \quad C \succ D \succ A \succ B. \tag{4}$$

Each candidate is in first, second, third, and fourth place precisely once, so all positional outcomes are ties. Moreover, each ranking is accompanied by its reversal, so all pairwise rankings are ties. But the situation changes with triplets; e.g., dropping D results in

$$A \succ B \succ C, \quad B \succ C \succ A, \quad B \succ A \succ C, \quad C \succ A \succ B \tag{5}$$

where, of all positional outcomes, only the Borda Count defines a complete tie so only it maintains consistency with the rankings of the other sets of candidates. As adding configurations of the Eq. 4 type to a profile can change the rankings of triplets, but affects nothing else, these configurations play a fundamental role in explaining Thm. 4 features where positional rankings of triplets differ from those of other subsets of candidates; it also helps explain why the Borda Count admits significantly fewer paradoxical outcomes than other positional rules.

To indicate how appropriate profile configurations are found, I'll outline the three-candidate structure. To do so, let me introduce a geometric way I developed to simplify the tallying of three-candidate ballots. As in Fig. 3, assign each candidate to a vertex of an equilateral triangle. Divide the triangle into regions based on the distance to each vertex where "closer is better;" e.g., as all points on the vertical line are equal distance to vertices A and B, this line corresponds to a tied $A \sim B$ ranking. Points to the left and right represent, respectively, $A \succ B$ and $B \succ A$.

The resulting 13 regions represent the 13 transitive rankings; the regions on lines correspond to rankings with ties.

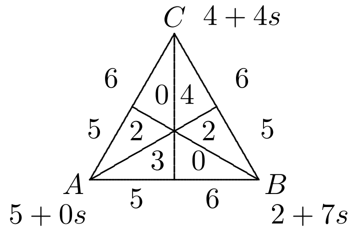


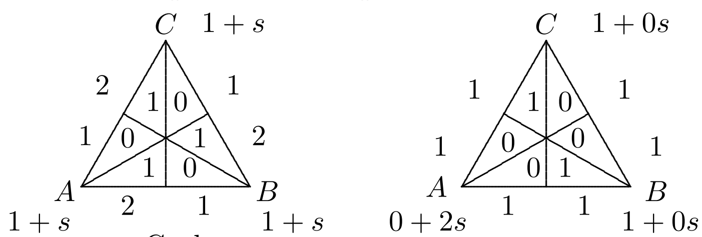
Figure 3. Geometric tally

In each open *ranking region*, place the number of the voters with this preference. Figure 3, for instance, has the introductory example data. To use the geometry to tally ballots, notice that all votes with $A \succ B$ are to the left of the vertical line. So, to compute the $\{A, B\}$ outcome, simply add the numbers on each side of the vertical line obtaining the $B \succ A$ outcome by $6 : 5$. The pairwise outcomes for each other pair are similarly computed and listed by the appropriate edge.

To compute all positional outcomes, normalize each $(w_1, w_2, 0)$ vector into a $\mathbf{w}_s = (1, s, 0)$ form, $s \in [0, 1]$, by dividing by w_1. This means, for instance, that the Borda $(2, 1, 0)$ vector is normalized to $(1, \frac{1}{2}, 0)$. The $\mathbf{w}_s$ tally for candidate C is

$$[\text{number of voters with } C \text{ top-ranked}] + s[\text{number of voters with } C \text{ second-ranked}].$$

Using the geometry, this tally is the sum of numbers in the two regions with C as a vertex plus s times the sum of numbers in the two adjacent regions, or $[4 + 0] + s[2 + 2]$. The $\mathbf{w}_s$ tallies for all three candidates are similarly computed and listed near the appropriate vertex on the triangle. In this simple manner, all positional and pairwise outcomes are easily computed.



a. Cycles **b.** Reversal

Figure 4. Symmetry structures

The geometry of tallying suggests we should examine the triangle's symmetry structure. A first choice is the *kernel* where each ranking is supported by the same number of voters; here all positional and pairwise outcomes are ties. The next obvious choice is the 120^o symmetries, or a Z_3 orbit defining, say, $A \succ B \succ C, B \succ C \succ A, C \succ A \succ B$. The tallies in Fig. 4a show that this configuration never affects positional rules as all outcomes are ties, but it does influence the pairs as they define a cycle. Thus, this kind of configuration in a profile causes the majority vote pairwise outcomes to differ from positional outcomes. A final symmetry is 180^o, or a Z_2 reversal such as $C \succ A \succ B$, $B \succ A \succ C$. As Fig. 4b shows, this configuration never affects pairs as they are all ties, but it affects all non-Borda positional outcomes, $s \neq \frac{1}{2}$, as they are not ties. Consequently, reversal configurations in a profile force differences among all non-Borda positional outcomes and differences from pairwise outcomes.

While a bit more work is required to convert all of this into a working tool for social scientists, such as creating a coordinate system for profile space [10], this structure explains all possible positional and pairwise three-candidate election inconsistencies. For instance, in the first paragraphs, I asked if the reader could explain why the introductory example allowed conflicting positional outcomes. As just indicated, the answer is that all possible differences among positional outcomes are caused by reversal configuration components in a profile. Indeed, to create the introductory example, I started with $C \succ B \succ A$ and added two reversal components of $A \succ C \succ B$, $B \succ C \succ A$ and three of $A \succ B \succ C$, $C \succ B \succ A$ to force the positional differences. While these terms influence the positional outcomes, they do not affect Borda or pairwise rankings, which remain $C \succ B \succ A$. If you wish to enhance the example with a cyclic pairwise outcome, just add an appropriate multiple of either the $A \succ B \succ C, B \succ C \succ A, C \succ A \succ B$ or the $A \succ C \succ B, C \succ B \succ A, B \succ A \succ C$ configuration; this term never affects positional rankings but it can change pairwise rankings. Notice, only the Borda Count is not affected by reversal and cyclic profile configurations; this turns out to be an explanation for the many favorable properties of the Borda Count and it explains why it is immune from many of the voting paradoxes.

The general n-candidate case is analyzed in a similar manner. Here, to use the geometry of tallying, the equilateral triangle is replaced with an equilateral n-simplex. Thus the symmetries of this simplex capture the structures of voting. A lesson learned from algebraic topology is how permutations in the interior of a simplex have an interesting effect on the faces; some of this is captured here by wreath products of permutation groups. Thus, a way to identify which profile configurations explain voting inconsistencies is to determine which orbits of subgroups create configurations that force completely tied outcomes for some subsets of candidates but not for others.

For a flavor of what happens in general, it turns out for n candidates that Z_n orbits of the

$$A_1 \succ A_2 \succ \ldots, \succ A_n, \quad A_2 \succ \ldots \succ A_n \succ A_1, \quad \ldots, \quad A_n \succ A_1 \succ \ldots \succ A_{n-1} \tag{6}$$

type explain all possible problems, inconsistencies, etc., of majority votes over pairs. Adding such a configuration to any profile does not affect n-candidate positional rankings, but it can change majority vote rankings with pairs. Such orbits, however, affect *all positional outcomes* for smaller subsets of candidates. (To see why, consider the $n = 4$ case and compute what happens when a candidate is dropped.)

Incidentally, the Eq. 6 profile configurations, which affect all positional rules, turns out to be the sole cause of inconsistent rankings for the Borda Count. As such, a strong case can be made that the Borda ranking over all candidates most accurately reflects the views of the voters. A stronger case comes from noting that certain profile configurations that are orbits of symmetry groups should result in ties, but only the Borda Count always respects this.

4 Other rules and extensions.

As this kind of symmetry structure answers questions about voting rules, it also answers questions about other aggregations methods. For instance, Anna Bargagliotti and I are developing related symmetry arguments to explain mysteries about non-parametric statistical rules. This structure also answers other voting mysteries. Arrow's Theorem and Sen's Theorem, for instance, have disturbing conclusions asserting that it is impossible for any voting rule to do what appears to be obviously possible to do. But by examining these results in the light of the above symmetry structures, it turns out [9, 10] that these seminal theorems occur because assumptions requiring the decision rule to emphasize pairs negate the crucial assumption that voters have transitive preferences. By understanding the mathematical source of the problem, benign resolutions are immediate.

There are all sorts of other voting concerns; e.g., voters might be strategic, or there may be too many candidates to realistically expect voters to rank them. But by understanding the mathematical structure of voting rules, all such issues can be addressed. A similar comment applies to wide classes of other voting rules; e.g., both the AMS and MAA use something called "Approval Voting" (AV) where a voter votes "approval" for as many candidates as he or she wishes. Stated in another manner, a voter ranks the candidates and selects which "vote-for-k" rule to tally the preferences. From Thm. 4, where it is shown how the "vote-for-k" rules allow so many problems, and the fact that the dimension of the associated domain increases significantly, it is easy to show that AV introduces many new, troubling problems. (The introductory example, for instance, allows all 13 ways to rank the candidates to be admissible AV outcomes.)

An appropriate concluding comment is to recall how for a couple of millennia, mathematics and the physical sciences have enjoyed a symbiotic relationship where advances in one area motivated advances in the other. There is a new opportunity; within the last couple of decades, the social and behavioral sciences have become mathematically more sophisticated, which suggests that a similar, mutually advantageous relationship can be developed. Beyond voting, areas that appear ripe for mathematical analysis include behavioral sciences such as psychology and social sciences such as economics and political science. What I find particularly attractive about these areas is that their issues differ from that found in the physical sciences, so new kinds of mathematics often is needed. In other words, the underlying mathematical structures needed to convert the somewhat ad-hoc mathematical analysis often being currently used into a systematic approach awaits some mathematician to develop them. I invite more mathematicians to examine these fascinating topics.

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