Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*

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*This essay is adapted, with permission, from Steven J. Brams, *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures* (Princeton University Press, 2008).
The book offers a detailed development of the ideas discussed herein and can be found at, http://press.princeton.edu/titles/8566.html.

Introduction

In *Mathematics and Democracy*, I try to show how mathematics can be used to illuminate two essential features of democracy:

- how individual preferences can be aggregated to give a social choice or election outcome that reflects the interests of the electorate; and
- how public and private goods can be divided in a way that respects due process and the rule of law.

Whereas questions of aggregation are the focus of *social choice theory*, questions of division are the focus of *fair division*.

Democracy, as I use the term, will generally mean representative democracy, in which citizens vote for representatives, from a president on down. But I also analyze referendums, in which citizens vote directly on propositions, just as they did in assemblies in ancient Greece.

I focus on *procedures*, or rules of play, that produce outcomes. By making precise properties that one wishes a voting or fair-division procedure to satisfy and clarifying relationships among these properties, mathematical analysis can strengthen the intellectual foundations on which democratic institutions are built. But because there may be no procedure or institution that satisfies all the properties one might desire, I examine trade-offs among the properties. In the case of some procedures, I also consider practical problems of implementation and discuss experience with those that have been tried out.

Institutional Design and Engineering

The voting and fair-division procedures I analyze foster democratic choices by giving voters better ways of expressing themselves, by electing officials who are more likely to be responsive to the electorate, and by allocating goods to citizens that ensure their shares are equitable or preclude envy. In some cases I criticize current procedures, but most of the analysis is constructive—I suggest how these procedures may be improved.

Designing procedures that satisfy desirable properties, or showing the limits of doing so, is sometimes referred to as *institutional design* or *mechanism design*. I present empirical examples to illustrate this approach, but the bulk of the analysis is theoretical.

The product of such analysis is normative: The prescription of new procedures or institutions that are superior, in terms of the criteria set forth, to ones that arose more haphazardly. Like engineering in the natural sciences, which translates theory (e.g., from physics) into practical design (e.g., a bridge), engineering in the social sciences translates theory into the design of political-economic-social institutions that better meet the criteria one deems important.

Mathematics and Democracy is divided into two parts:

Part 1. Voting Procedures

One cornerstone of democracy is honest and periodic elections, wherein there is meaningful competition among parties, interest groups, or individuals for political office. Several of the voting procedures that are analyzed are relatively new and not well known, but they offer significant advantages over extant procedures. Common to many of them is approval balloting, whereby voters can approve of as many candidates of alternatives as they like without having to rank them.

Approval balloting may take different forms. Under approval voting, the candidate or alternative with the most votes wins. Under other methods of aggregating approval votes, different candidates or alternatives may win. These methods maximize different objective functions, or constrain outcomes in certain ways, in order to achieve certain ends, such as the proportional representation of different interests in the electorate.

Most social-choice analysis assumes rational individuals, who select the most effective or efficient means to satisfy their goals, and examines the implications of these individual choices on collective choices. *Game theory* is an important tool in such analysis, especially in identifying outcomes that are stable or in equilibrium, and institutions that support the equilibria one finds. These institutions do not necessarily dispel conflict but manage it so that political life continues, with politicians never being sure that they will continue in office but quite sure that the institutions will persist.

Part 2. Fair-Division Procedures

As central as elections are to the performance of a democracy, a democracy would be a sham if the politicians elected were not restricted by due process and the rule of law. Ideally, democracies treat all citizens the same way—at least when governed by a constitution or other laws—particularly with respect to their civil rights and certain freedoms, such as freedom of association and freedom of religion.

The equal treatment of citizens depends in part on their receiving fair shares of things that must be divided among them, not just on the efficiency of outcomes. Accordingly, I analyze different procedures of fair division—applicable to both divisible and indivisible goods—and study their distributional consequences.

Fairness requires that one take into account the different preferences or claims of players who have a stake in an outcome. Step-by-step rules or algorithms to implement the fair division of goods, which may be homogenous (like money) or heterogeneous (like land with different objects on it), are analyzed. Questions that relate to the fair division of people or groups include: What political parties are best suited to form a government? Which parties should get what cabinet ministries in the government?

Use of Mathematics and Scope

While a good mathematical background makes reading the theoretical parts of *Mathematics and Democracy* easier, several chapters are accessible to those with little mathematical training. When a topic in a chapter goes beyond the level of the rest of the chapter or is a digression from its main theme, I discuss it in an appendix to the chapter.

Of course, some chapters are inherently more analytic or mathematical than others, so the reader may want to skip those that cause difficulty. In fact, I encourage selective reading of the chapters, most of which are relatively self-contained and can be read independently of others. A glossary at the end provides a quick reference to the most important concepts that I use in the book.

I have certainly not covered all institutions in the public sphere. For example, there is now a large literature on redistricting, or the drawing of district boundaries after a census; on auctions, which governments employ to sell such things as oil leases and parts of the electronic spectrum; and on matching algorithms, which are used in the selection of schools by children and parents, and hospital residencies by doctors. There is also a substantial qualitative literature on problems of implementing and evaluating democratic reforms.

Conclusions of the Study

Because there is not one voting or fair-division procedure that fits all circumstances, I have proposed and analyzed several of each. Although many of the procedures are not well-known, each meets important needs in the diverse situations described.

What may surprise theorists of democracy is the degree to which the justifications of the procedures depend on mathematical analysis. Although commonsensical arguments can be made for most of them, precisely what properties each satisfies (or does not satisfy) require careful and rigorous analysis that is fundamentally mathematical in nature.

But as I show in some cases, there may be no procedure that guarantees all the properties one might want in a procedure, especially when one is allocating indivisible goods. In this situation, I think it important to understand the trade-offs that must be made. Thereby a better informed choice can be made of which properties to give priority to when all cannot, even in principle, be met.

The fair-division procedures, especially, may seem quite remote from what is needed to makes a democracy run well. But recall that once candidates are elected, their policies are often ill-formed, leading to arbitrary or inchoate choices that seem unfair to many citizens. In such a situation, all the good will in the world will not be enough to repair the damage caused.

Politicians' decisions about how to allocate goods need to be informed by more than rhetoric and vague notions of fairness. If they are not, the ad hoc decisions they make will be plagued by inconsistency and are more likely to cause anger and strife than satisfy the electorate. Worse, if the procedures are highly manipulable, they are likely to be undermined or corrupted by self-seeking (yet rational) politicians.

This is why I emphasize the stability of the outcomes the procedures produce and the strategy-proofness of some of the procedures. If it is not rational to be manipulative, outcomes are more likely to be responsive to all players' preferences and, consequently, accepted as legitimate.

Despite the plethora of procedures I discuss, there are surely new ones to be discovered, and old ones to be resurrected and rehabilitated, that will foster more robust democratic institutions. I urge political scientists, mathematicians, and other scholars to continue the search for these.

But I also exhort those with a serious practical interest in making democracy work better to lend aid and encouragement, especially in helping to implement the theoretically most compelling procedures to test whether they work well in practice. These tests will inspire new theoretical advances, bringing the scientific enterprise full circle in enhancing democratic institutions.

Case Study: Approval Voting

It may come as a surprise to some that there is a science of elections, whose provenance can be traced back to the Marquis de Condorcet in 18^{th}-century France, Charles Dodgson (Lewis Carroll) in 19^{th}-century England, and Kenneth Arrow in 20^{th}-century America. Since Arrow published his seminal book, *Social Choice and Individual Values*, in 1951—for which in large part he received the Nobel Memorial Prize in Economics in 1972—there have been thousands of articles and hundreds of books published on everything from the mathematical properties of voting systems to empirical tests of the propensity of different systems to elect centrist candidates.

The 2000 US presidential election highlighted, among other things, the frailties of voting machines and the seeming arbitrariness of such venerable US institutions as the Electoral College and the Supreme Court. Political commentary has focused on these aspects but given very little attention to alternative voting systems, about which the science of elections has much to say.

Several alternative systems for electing a single winner have been shown to be far superior to plurality voting (PV)—the most common voting system used in the United States as well as many other places—in terms of a number of criteria. PV, which allows citizens to vote for only one candidate, suffers from a dismaying flaw. In any race with more than two candidates, PV may elect the candidate least acceptable to the majority of voters. This frequently happens in a three-way contest, when the majority splits its votes between two centrist candidates. PV also forces minor-party candidates into the role of spoilers—as we seen in the 2000 presidential election with the candidacy of Ralph Nader, who received only 2.7 percent of the popular vote—which can be decisive in a close contest between the major-party candidates.

Of the alternatives to PV, I recommend approval voting (AV), on both practical and theoretical grounds, in single-winner elections. Proposed independently by several analysts in the l970s, AV is a voting procedure in which voters can vote for, or approve of, as many candidates as they wish in multicandidate elections—that is, elections with more than two candidates. Each candidate approved of receives one vote, and the candidate with the most votes wins. AV is now used by several professional societies, including the American Mathematical Society, the Mathematical Association of America, and the American Statistical Association to elect officers.

In the United States, the case for AV seems particularly strong in primary and nonpartisan elections, which often draw large fields of candidates. Here are some commonsensical arguments for AV that have been made:

1. *It gives voters more flexible options*. They can do everything they can under PV—vote for a single favorite—but if they have no strong preference for one candidate, they can express this fact by voting for all candidates they find acceptable. In addition, if a voter's most preferred candidate has little chance of winning, that voter can vote for both a first choice *and* a more viable candidate without worrying about wasting his or her vote on the less popular candidate.

2. *It helps elect the strongest candidate*. Today the candidate supported by the largest minority often wins, or at least makes the runoff if there is one. Under AV, by contrast, the candidate with the greatest overall support will generally win. In particular, *Condorcet winners*, who can defeat every other candidate in separate pairwise contests, almost always win under AV, whereas under PV they often lose because they split the vote with one or more other centrist candidates.

3. *It will reduce negative campaigning.* AV induces candidates to try to mirror the views of a majority of voters, not just cater to minorities whose voters could give them a slight edge in a crowded plurality contest. It is thus likely to cut down on negative campaigning, because candidates will have an incentive to try to broaden their appeals by reaching out for approval to voters who might have a different first choice. Lambasting such a choice would risk alienating this candidate's supporters and losing their approval.

4. *It will increase voter turnout*. By being better able to express their preferences, voters are more likely to vote in the first place. Voters who think they might be wasting their votes, or who cannot decide which of several candidates best represents their views,

will not have to despair about making a choice. By not being forced to make a single—perhaps arbitrary—choice, they will feel that the election system allows them to be more honest, which will make voting more meaningful and encourage greater participation in elections.

5. *It will give minority candidates their proper due*. Minority candidates will not suffer under AV: Their supporters will not be torn away simply because there is another candidate who, though less appealing to them, is generally considered a stronger contender. Because AV allows these supporters to vote for *both* candidates, they will not be tempted to desert the one who is weak in the polls, as under PV. Hence, minority candidates will receive their true level of support under AV, even if they cannot win. This will make election returns a better reflection of the overall acceptability of candidates, relatively undistorted by strategic voting, which is important information often denied to voters today.

6. *It is eminently practicable*. Unlike more complicated ranking systems, which suffer from a variety of theoretical as well as practical defects, AV is simple for voters to understand and use. Although more votes must be tallied under AV than under PV, AV can readily be implemented on existing voting machines. Because AV does not violate any state constitutions in the United States (or, for that matter, the constitutions of most countries in the world), it requires only an ordinary statute to enact.

Voting systems that involve ranking candidates may appear, at first blush, more appealing than AV. One, the Borda count, awards points to candidates according to their ranking. Another is the Hare system of single transferable vote (STV)—with variants called the “alternative vote” and “instant runoff”—in which candidates receiving the fewest first-choice votes are progressively eliminated. Their votes are transferred to second choices—and lower choices if necessary—until one candidate emerges with a majority of voters.

Compared with AV, these systems have serious drawbacks. The Borda count fosters "insincere voting" (for example, ranking a second choice at the bottom if that candidate is considered the strongest threat to one's top choice) and is also vulnerable to "irrelevant candidates" who cannot win but can affect the outcome. STV may eliminate a centrist candidate early on and thereby elect one less acceptable to the majority. It also suffers from "nonmonotonicity," in which voters, by raising the ranking of a candidate, may actually cause that candidate to lose—just the opposite of what one would want to happen.

As cherished a principle as "one person, one vote" is in single-winner elections, democracies, I believe, can more benefit from the alternative principle of "one candidate, one vote," whereby voters make judgments about whether each candidate on the ballot is acceptable or not. The latter principle makes the tie-in of a vote not to the voter but rather to the candidates, which is arguably more egalitarian than artificially restricting voters to casting only one vote in multicandidate races. This principle also affords voters an opportunity to express their intensities of preference by approving of, for example, all candidates except the one they despise.

Although AV encourages sincere voting, it does not altogether eliminate strategic calculations. Because approval of a less-preferred candidate can hurt a more-preferred approved candidate, the voter is still faced with the decision of where to draw the line between acceptable and unacceptable candidates. A rational voter will vote for a second choice if his or her first choice appears to be a long shot—as indicated, for example, by polls—but the voter's calculus and its effects on outcomes is not yet well understood for either AV or other voting procedures.

While AV is a strikingly simple election reform for finding consensus choices in single-winner elections, in elections with more than one winner—such as for a council or a legislature—AV would not be desirable if the goal is to mirror a diversity of views, especially of minorities; for this purpose, other voting systems should may be more desirable. On the other hand, minorities may derive indirect benefit from AV in single-winner elections, because mainstream candidates, in order to win, will be forced to reach out to minority voters for the approval *they* (the mainstream candidates) need in order to win. While promoting majoritarian candidates, therefore, AV induces them to be responsive to minority views.